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Emmanuel Frénod

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E. Frénod*

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Abstract: we show that the solution to an oscillatory-singularly perturbed ordinary differential equation may be asymptotically expanded into a sum of oscillating terms. Each of those terms writes as an oscillating operator acting on the solution to a non oscillating ordinary differential equation with an oscillating correction added to it.
The expression of the non oscillating ordinary differential equations are defined by a recurrence relation.
We then apply this result to problems where charged particles are submitted to large magnetic field.

1 Introduction and results

Purpose
The goal of this paper is to manage and justify the asymptotic two scale expansion as $\varepsilon \to 0$ of the solution $\mathbf{X}_\varepsilon(t) = \mathbf{X}_\varepsilon(t; \mathbf{x}, s)$ to the following singularly perturbed dynamical system:

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{a}(t, \frac{t-s}{\varepsilon}, \mathbf{X}_\varepsilon) + \frac{1}{\varepsilon}\mathbf{b}(t, \mathbf{X}_\varepsilon), \quad \mathbf{X}_\varepsilon(s; \mathbf{x}, s) = \mathbf{x}, \tag{1.1}$$

and to apply this to models of charge particles submitted to a strong magnetic field which is possibly non uniform.

More precisely, C_b^m standing for the space of functions that have continuous and bounded derivatives until the order m, we assume that

$$\mathbf{a}(\cdot, \cdot, \cdot) \in (C_b^{m+1}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d))^d, \ \theta \mapsto \mathbf{a}(t, \theta, \mathbf{x}) \text{ is } 2\pi\text{-periodic for every } t \in \mathbb{R} \text{ and } \mathbf{x} \in \mathbb{R}^d, \tag{1.2}$$

$$\mathbf{b}(\cdot, \cdot) \in (C_b^{m+2}(\mathbb{R} \times \mathbb{R}^d))^d, \tag{1.3}$$

for some $m \geq 0$. In particular, it implies that $|\nabla_x \cdot \mathbf{b}| \leq C$, uniformly on $\mathbb{R} \times \mathbb{R}^d$. We also suppose that the solution $\mathbf{Z}(t, \theta; \mathbf{z})$ to

$$\frac{\partial \mathbf{Z}}{\partial \theta} = \mathbf{b}(t, \mathbf{Z}), \quad \mathbf{Z}(t, 0; \mathbf{z}) = \mathbf{z}, \tag{1.4}$$

is known and $2\pi-$periodic in θ for every $t \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^d$.

Under those assumptions, we prove that $\mathbf{X}_\varepsilon(\cdot; \mathbf{x}, s) : \mathbb{R} \to \mathbb{R}^d$ admits the following expansion:

$$\mathbf{X}_\varepsilon(t; \mathbf{x}, s) = \mathbf{X}^0(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s) + \varepsilon \mathbf{X}^1(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s) + \varepsilon^2 \mathbf{X}^2(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s) + \dots, \tag{1.5}$$

*LMAM, Université de Bretagne Sud, Centre Yves Coppens, Campus de Tohannic, F-56000, Vannes

as $\varepsilon \to 0$, where the $\mathbf{X}^i(t,\theta;\mathbf{x},s)$ are $2\pi-$periodic in θ. The initial condition in (1.1) is assigned to $\mathbf{X}^0$ by:

$$\mathbf{X}^0(s,0;\mathbf{x},s) = \mathbf{x} \text{ and } \mathbf{X}^i(s,0;\mathbf{x},s) = 0 \text{ for } i \geq 1. \tag{1.6}$$

Motivations

The target we have in mind while writing this paper is the conception of pic-methods for plasmas submitted to large magnetic field. Direct applications of this are the simulation of magnetic confinement fusion or isotope resonant separation experiments.

A plasma submitted to a strong magnetic field could be modelled by several Vlasov equations, according to experimental conditions. For instance if we are interested in describing the global behaviour of a plasma in a magnetic confinement fusion experiment, the following equation

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}\cdot\nabla_x f^\varepsilon + \Big((\mathbf{E}(t,\mathbf{x}) + \frac{\mathcal{N}}{\varepsilon}) + \mathbf{v}\times(\mathbf{B}(t,\mathbf{x}) + \frac{\mathcal{M}}{\varepsilon})\Big)\cdot\nabla_v f^\varepsilon = 0, \quad f^\varepsilon_{|t=s} = f_0, \tag{1.7}$$

for a small parameter ε would be a relevant model equation. In equation (1.7), $f^\varepsilon(t,\mathbf{x},\mathbf{v})$ is at time t, the particle density of the plasma in position $\mathbf{x}$ and with velocity $\mathbf{v}$. The vector $\mathcal{M}\in\mathbb{S}^2$ gives the direction of the strong magnetic field. This direction may depend on t and $\mathbf{x}$. the vector $\mathcal{N}\in\mathbb{S}^2$ with $\mathcal{N}\perp\mathcal{M}$ is the direction of a strong electric field ($\mathcal{N}$ may also be 0). The vector fields $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields that may contain a self consistent part. We shall call the scaling leading to this equation the "Guiding Centre Regime".

If now we are interested in understanding what happens close to the plasma boundary in a tokamak, we would prefer to use the so called "Finite Larmor Radius Regime". In other words, we would consider an equation of the type:

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}_\parallel\cdot\nabla_x f^\varepsilon + \frac{\mathbf{v}_\perp}{\varepsilon}\cdot\nabla_x f^\varepsilon + \Big(\mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\frac{\mathcal{M}}{\varepsilon}\Big)\cdot\nabla_v f^\varepsilon = 0, \quad f^\varepsilon_{|t=s} = f_0, \tag{1.8}$$

where the meaning of $f^\varepsilon(t,\mathbf{x},\mathbf{v})$ is the same as above and where $\mathbf{v}_\parallel = (\mathbf{v}\cdot\mathcal{M})\mathcal{M}$, $\mathbf{v}_\perp = \mathbf{v}-\mathbf{v}_\parallel$.

Lastly, for isotope resonant separation we would be interested in regarding

$$\frac{\partial f^\varepsilon}{\partial t} + \mathbf{v}\cdot\nabla_x f^\varepsilon + \Big(\mathbf{E}(t,\frac{t-s}{\varepsilon},\mathbf{x}) + \mathbf{v}\times(\mathbf{B}(t,\frac{t-s}{\varepsilon},\mathbf{x}) + \frac{\mathcal{M}}{\varepsilon})\Big)\cdot\nabla_v f^\varepsilon = 0, \quad f^\varepsilon_{|t=s} = f_0, \tag{1.9}$$

where the electric and magnetic fields $\mathbf{E}(t,\frac{t-s}{\varepsilon},\mathbf{x})$ and $\mathbf{B}(t,\frac{t-s}{\varepsilon},\mathbf{x})$ oscillate with the same frequency $1/(2\pi\varepsilon)$ as the cyclotron frequency of the plasma particles.

We refer to Frénod and Sonnendrücker [10, 11, 12, 13, 14], Frénod, Raviart and Sonnendrücker [9] and Frénod and Watbled [15] where those models are explained and asymptotically analysed. In particular, they give indication on how to get the self consistent part of the electric field in some examples and under suitable assumptions. Complementary works concerning the asymptotic behaviour of those kind of equations are also led in Golse and Saint Raymond [16, 17], Saint Raymond [32], Brenier [4], Grenier [19, 20], Jabin [21], Schochet [35], Joly, Métivier and Rauch [24]. We also refer to mathematical or physical works where similar methods are used: [22, 23, 36, 7, 8, 27, 26, 6, 5, 28, 18].

It is relatively easy to see that equations (1.7), (1.8) and (1.9) enter the framework of a singularly perturbed convection equation reading:

$$\frac{\partial u_\varepsilon}{\partial t} + \mathbf{a}\cdot\nabla u_\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u_\varepsilon = 0, \qquad u_{\varepsilon|t=s} = u_0, \tag{1.10}$$

for $\mathbf{x}\in\mathbb{R}^d$ and $t > s$. At least formally, the solution $u_\varepsilon(t,\mathbf{x})$ is given by

$$u_\varepsilon(t,\mathbf{x}) = u_0(\mathbf{X}_\varepsilon(s;\mathbf{x},t)). \tag{1.11}$$

Numerical methods to solve equation (1.7), (1.8) or (1.9), or similar ones, will be applications of generic methods solving (1.10). Since u_ε contains high frequency oscillations, using in state any well known numerical method to solve (1.10) would compel to use a very small time step.

Yet, in order to relax this constraint, we may follow, at least, two strategies. The first one could be based on the work presented in Frénod, Raviart and Sonnendrücker [9] where we proved that u_ε writes $u_\varepsilon(t,\mathbf{x}) = \sum_{i\geq 0} \varepsilon^i U^i(t, \frac{t-s}{\varepsilon}, \mathbf{x})$, and where we determined the equations satisfied by every terms U^i. Since those equations are independent of ε, the computation of the first terms U^i could be led using a standard numerical method. Then an approximation of u_ε would be given by: $u_\varepsilon(t,\mathbf{x}) \simeq \sum_{i=0}^k \varepsilon^i U^i(t, \frac{t-s}{\varepsilon}, \mathbf{x})$, for some $k \in \mathbb{N}$.

The second strategy consists in using an approximation $\mathbf{X}_\varepsilon(t;\mathbf{x},s) \simeq \sum_{i=0}^k \varepsilon^i \mathbf{X}^i(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s)$, of (1.5), for some $k \in \mathbb{N}$, after computing $\mathbf{X}^0, \mathbf{X}^1, \dots, \mathbf{X}^k$. Then the expression (1.11) yields an approximation of u_ε: $u_\varepsilon(t,\mathbf{x}) \simeq u_0\big(\sum_{i=0}^k \varepsilon^i \mathbf{X}^i(s, -\frac{t-s}{\varepsilon}; \mathbf{x}, t)\big)$. This motivates the present work.

We give now some references concerning perturbed ordinary differential equations. First, we think to the Lindstedt-Poincaré method explained in Poincaré [31] where the steady state periodic solutions to a perturbed second order ordinary differential equation is studied. Then the Krylov-Bogoliubov-Mitropolsky method, see [3] and [25], allows to describe the transitory behaviour of the solution to a perturbed ordinary differential equation to a periodic solution.

We also cite the works of Verhulst [42] and Sanders and Verhulst [34] (see also Mickens [29]) where the Method of Averaging is developed to treat perturbed ordinary differential equations and adiabatic invariants in Hamiltonian systems.

Lastly, we mention the work initiated by Tikhonov [38, 39, 40] and developed by Vasilieva and others (see the review paper of Vasilieva [41] and the references in it).

This work is also related to homogenisation methods based on weak$-*$ convergence and two scale convergence. The most important ideas about those tools may be found in Tartar [37], Bensoussan, Lions and Papanicolaou [2], Sanchez-Palencia [33], N'Guetseng [30], Allaire [1] and Frénod, Raviart and Sonnendrücker [9].

Theorems

We first justify the expansion (1.5) until order 0.

Theorem 1.1 *Setting*

$$\tilde{\mathbf{a}}^0(t,\mathbf{y}^0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{\alpha}^0(t,\theta,\mathbf{y}^0)\,d\theta, \tag{1.12}$$

with

$$\tilde{\alpha}^0(t,\theta,\mathbf{y}^0) = \big\{\nabla_z \mathbf{Z}(t,\theta;\mathbf{y}^0)\big\}^{-1}\big\{\mathbf{a}(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0)) - \frac{\partial \mathbf{Z}}{\partial t}(t,\theta;\mathbf{y}^0)\big\}, \tag{1.13}$$

under assumptions (1.2), (1.3) with $m=0$ and (1.4), for any $\mathbf{x} \in \mathbb{R}^d$, $s \in \mathbb{R}$, $T \in \mathbb{R}$ and any $\varepsilon > 0$, the solution $\mathbf{X}_\varepsilon(\cdot;\mathbf{x},s)$ of (1.1) exists on $[s, s+T]$, is unique and the sequence $(\mathbf{X}_\varepsilon(\cdot;\mathbf{x},s))$ satisfies:

$$\lim_{\varepsilon\to 0} \sup_{t\in[s,s+T]} \big|\mathbf{X}_\varepsilon(t;\mathbf{x},s) - \mathbf{X}^0(t, \frac{t-s}{\varepsilon}; \mathbf{x}, s)\big| = 0, \tag{1.14}$$

$|\cdot|$ standing for the Euclidean norm on $\mathbb{R}^d$, where $\mathbf{X}^0$ satisfies

$$\mathbf{X}^0(t,\theta;\mathbf{x},s) = \mathbf{Z}(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s)), \tag{1.15}$$

and where $\mathbf{Y}^0$ is the solution to

$$\frac{d\mathbf{Y}^0}{dt} = \tilde{\mathbf{a}}^0(t,\mathbf{Y}^0), \quad \mathbf{Y}^0(s;\mathbf{x},s) = \mathbf{x}. \tag{1.16}$$

In the Theorem above, $\nabla_z \mathbf{Z}(t,\theta;\mathbf{z})$ stands for the Jacobian matrix of $\mathbf{z} \mapsto \mathbf{Z}(t,\theta;\mathbf{z})$.

In order to set this Theorem and to justify the expansion (1.5) for higher orders, using a Van der Pol transformation, we define $\mathbf{Y}_\varepsilon$ being such that

$$\mathbf{X}_\varepsilon(t;\mathbf{x},s) = \mathbf{Z}(t, \frac{t-s}{\varepsilon}; \mathbf{Y}_\varepsilon(t,\mathbf{x},s)) = [\mathbf{Z}(\mathbf{Y}_\varepsilon)]_\varepsilon, \tag{1.17}$$

where for any function f we write $[f(\mathbf{Y}_\varepsilon,\dots,\mathbf{Y}_\varepsilon^k)]_\varepsilon$ for $f(t, \frac{t-s}{\varepsilon}, \mathbf{Y}_\varepsilon(t;\mathbf{x},s),\dots,\mathbf{Y}_\varepsilon^k(t;\mathbf{x},s))$ and $[f(\mathbf{Y}^0,\dots,\mathbf{Y}^k)]_\varepsilon$ for $f(t,\frac{t-s}{\varepsilon},\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^k(t;\mathbf{x},s))$. It is an easy game to show that

$$\frac{d\mathbf{Y}_\varepsilon}{dt} = \tilde{\alpha}^0(t,\frac{t-s}{\varepsilon},\mathbf{Y}_\varepsilon) = [\tilde{\alpha}^0(\mathbf{Y}_\varepsilon)]_\varepsilon; \;\; \mathbf{Y}_\varepsilon(s;\mathbf{x},s) = \mathbf{x}. \tag{1.18}$$

Indeed, derivating (1.17) and using (1.18) we get

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \frac{1}{\varepsilon}\big[\frac{\partial \mathbf{Z}}{\partial\theta}(\mathbf{Y}_\varepsilon)\big]_\varepsilon + \big[\frac{\partial \mathbf{Z}}{\partial t}(\mathbf{Y}_\varepsilon)\big]_\varepsilon + \big\{\nabla_z \mathbf{Z}(\mathbf{Y}_\varepsilon)\}_\varepsilon\{\tilde{\alpha}^0(\mathbf{Y}_\varepsilon)\big\}_\varepsilon = [\mathbf{a}(\mathbf{X}_\varepsilon)]_\varepsilon + \frac{1}{\varepsilon}\mathbf{b}(\mathbf{X}_\varepsilon), \tag{1.19}$$

and we have the initial condition $\mathbf{X}_\varepsilon(s;\mathbf{x},s) = \mathbf{x}$.
We also use the following notation. For a vector field $\mathbf{Z}$, The $i-$th component of $\{\nabla_x^k \mathbf{Z}\}\{\mathbf{x}^0,\dots,\mathbf{x}^k\}$, for $i=1,\dots,d$, is given by:

$$\big(\{\nabla_x^k \mathbf{Z}\}\{\mathbf{x}^0,\mathbf{x}^1,\dots,\mathbf{x}^k\}\big)_i = \sum_{l_1,\dots,l_k=1}^{d} \frac{\partial^k \mathbf{Z}_i}{\partial x_{l_1}\dots\partial x_{l_k}} \mathbf{x}_{l_1}^0 \dots \mathbf{x}_{l_k}^k. \tag{1.20}$$

In order to simplify we shall sometimes denote $\{\nabla_x^k \mathbf{Z}\}\{\mathbf{x}^0,\mathbf{x}^0,\dots,\mathbf{x}^0\}$ by $\{\nabla_x^k \mathbf{Z}\}\{\mathbf{x}^0\}^k$.

Now for $k\geq 0$ we recursively define

$$\tilde{\mathbf{A}}^k(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k) = \frac{1}{\theta}\int_0^\theta \tilde{\alpha}^k(t,\sigma,\mathbf{y}^0,\dots,\mathbf{y}^k)\,d\sigma - \tilde{\mathbf{a}}^k(t,\mathbf{y}^0,\dots,\mathbf{y}^k), \tag{1.21}$$

where $\tilde{\alpha}^0$ and $\tilde{\mathbf{a}}^0$ are given by (1.13) and (1.12) and where for $k\geq 1$ we have:

$$\begin{aligned}
&\tilde{\alpha}^k(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k) = \big\{\nabla_{y^0}\tilde{\alpha}^0(t,\theta;\mathbf{y}^0)\big\}\{\mathbf{y}^k + \theta\tilde{\mathbf{A}}^{k-1}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{k-1})\}+\\
&\frac{1}{2}\big\{\nabla_{y^0}^2\tilde{\alpha}^0(t,\theta;\mathbf{y}^0)\big\}\Big(\sum_{j=1}^{k-1}\{\mathbf{y}^j + \theta\tilde{\mathbf{A}}^{j-1}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{j-1})\,,\,\mathbf{y}^{k-j} + \theta\tilde{\mathbf{A}}^{k-j-1}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{k-j-1})\}\Big)+\\
&\dots + \frac{1}{k!}\big\{\nabla_{y^0}^k\tilde{\alpha}^0(t,\theta;\mathbf{y}^0)\big\}\{\mathbf{y}^1+\theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0)\}^k-\\
&\qquad\theta\Big(\sum_{j=0}^{k-1}\{\nabla_{y^j}\tilde{\mathbf{A}}^{k-1}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{k-1})\}\{\tilde{\mathbf{a}}^j(t,\mathbf{y}^0,\dots,\mathbf{y}^j)\} + \frac{\partial\tilde{\mathbf{A}}^{k-1}}{\partial t}(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^{k-1})\Big),
\end{aligned} \tag{1.22}$$

and

$$\tilde{\mathbf{a}}^k(t,\mathbf{y}^0,\dots,\mathbf{y}^k) = \frac{1}{2\pi}\int_0^{2\pi}\tilde{\alpha}^k(t,\theta,\mathbf{y}^0,\dots,\mathbf{y}^k)\,d\theta. \tag{1.23}$$

With those notations, we can state the asymptotic expansion of $\mathbf{Y}_\varepsilon$,

$$\mathbf{Y}_\varepsilon = \mathbf{Y}^0 + \varepsilon(\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon) + \varepsilon^2(\mathbf{Y}^2 + [\theta\tilde{\mathbf{A}}^1]_\varepsilon) + \dots. \tag{1.24}$$

In other words, defining for $k\geq 1$,

$$\begin{aligned}
\mathbf{Y}_\varepsilon^k &= \frac{1}{\varepsilon^k}\big(\mathbf{Y}_\varepsilon - \mathbf{Y}^0 - \varepsilon(\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon) - \dots - \varepsilon^{k-1}(\mathbf{Y}^{k-1} + [\theta\tilde{\mathbf{A}}^{k-2}]_\varepsilon)\big) - [\theta\tilde{\mathbf{A}}^{k-1}]_\varepsilon,\\
&= \frac{1}{\varepsilon}\big(\mathbf{Y}_\varepsilon^{k-1} - \mathbf{Y}^{k-1}\big) - [\theta\tilde{\mathbf{A}}^{k-1}]_\varepsilon,
\end{aligned} \tag{1.25}$$

we have the following Theorem.

THEOREM 1.2 *Under assumptions (1.2), (1.3) and (1.4) for any* $\mathbf{x} \in \mathbb{R}^d$, $s \in \mathbb{R}$ *and* $T \in \mathbb{R}$, *the sequences* $(\mathbf{Y}_\varepsilon^k(\cdot;\mathbf{x},s))$, *for* $k = 0,\dots,m$, *are bounded in* $L^\infty([s,s+T])$ *and we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in[s,s+T]}\big|\mathbf{Y}_\varepsilon(t;\mathbf{x},s) - \mathbf{Y}^0(t;\mathbf{x},s)\big| = 0, \ \ and \ \lim_{\varepsilon\to 0}\sup_{t\in[s,s+T]}\big|\mathbf{Y}_\varepsilon^k(t;\mathbf{x},s) - \mathbf{Y}^k(t;\mathbf{x},s)\big| = 0, \tag{1.26}$$

for $k \geq 1$, *where* $\mathbf{Y}^0$ *is solution to (1.16) and where* $\mathbf{Y}^k$ *is the solution to*

$$\frac{d\mathbf{Y}^k}{dt} = \tilde{\mathbf{a}}^k(t,\mathbf{Y}^0,\dots,\mathbf{Y}^k), \ \ \mathbf{Y}^k(s;\mathbf{x},s) = 0. \tag{1.27}$$

As a consequence of this Theorem and of (1.17), defining $(\mathbf{X}_\varepsilon^k(t;\mathbf{x},s))$ by:

$$\mathbf{X}_\varepsilon^k = \frac{1}{\varepsilon^k}\big(\mathbf{X}_\varepsilon - [\mathbf{X}^0]_\varepsilon - \cdots - \varepsilon^{k-1}[\mathbf{X}^{k-1}]_\varepsilon\big) = \frac{1}{\varepsilon}\big(\mathbf{X}_\varepsilon^{k-1} - [\mathbf{X}^{k-1}]_\varepsilon\big), \tag{1.28}$$

for $k \geq 1$, we have the rigorous justification of asymptotic expansion (1.5).

THEOREM 1.3 *Under assumptions (1.2), (1.3) and (1.4) for any* $\mathbf{x} \in \mathbb{R}^d$, $s \in \mathbb{R}$ *and* $T \in \mathbb{R}$, *the sequences* $(\mathbf{X}_\varepsilon^k(\cdot;\mathbf{x},s))$, *for* $k = 1,\dots,m$, *are bounded in* $L^\infty([s,s+T])$ *and we have:*

$$\lim_{\varepsilon\to 0}\sup_{t\in[s,s+T]}\big|\mathbf{X}_\varepsilon^k(t;\mathbf{x},s) - \mathbf{X}^k(t,\frac{t-s}{\varepsilon};\mathbf{x},s)\big| = 0, \tag{1.29}$$

where $\mathbf{X}^k$ *writes*

$$\begin{aligned}
&\mathbf{X}^k(t,\theta;\mathbf{x},s) = \big\{\nabla_z\mathbf{Z}(t,\theta;\mathbf{Y}^0(t;\mathbf{x},s))\big\}\big\{\mathbf{Y}^k(t;\mathbf{x},s) + \theta\tilde{\mathbf{A}}^{k-1}(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^{k-1}(t;\mathbf{x},s))\big\}+\\
&\frac{1}{2}\big\{\nabla_z^2\mathbf{Z}(t,\theta;\mathbf{Y}^0(t;\mathbf{x},s))\big\}\Big(\sum_{j=1}^{k-1}\big\{\mathbf{Y}^j(t;\mathbf{x},s) + \theta\tilde{\mathbf{A}}^{j-1}(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^{j-1}(t;\mathbf{x},s))\,,\\
&\qquad\mathbf{Y}^{k-j}(t;\mathbf{x},s) + \theta\tilde{\mathbf{A}}^{k-j-1}(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s),\dots,\mathbf{Y}^{k-j-1}(t;\mathbf{x},s))\big\}\Big)+\\
&\qquad\cdots + \frac{1}{k!}\big\{\nabla_z^k\mathbf{Z}(t,\theta;\mathbf{Y}^0(t;\mathbf{x},s))\big\}\{\mathbf{Y}^1(t;\mathbf{x},s) + \theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{Y}^0(t;\mathbf{x},s))\}^k,
\end{aligned} \tag{1.30}$$

with $\mathbf{Y}^k$ *solution to (1.27).*

We end this result list by giving explicit expressions. Using (1.13) in (1.22) we have the following expression of $\tilde{\alpha}^1$:

$$\begin{aligned}
&\tilde{\alpha}^1(t,\theta,\mathbf{y}^0,\mathbf{y}^1) = \big\{\nabla_z\mathbf{Z}(t,\theta;\mathbf{y}^0)\big\}^{-1}\Big\{\big\{\nabla_x\mathbf{a}(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0))\big\}\big\{\nabla_z\mathbf{Z}(t,\theta;\mathbf{y}^0)\big\}\big\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0(t,\theta;\mathbf{y}^0)\big\}-\\
&\big\{\frac{\partial\nabla_z\mathbf{Z}}{\partial t}(t,\theta;\mathbf{y}^0)\big\}\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0)\} - \big\{\nabla_z^2\mathbf{Z}(t,\theta;\mathbf{y}^0)\big\}\{\tilde{\alpha}^0(t,\mathbf{y}^0),\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0)\}\Big\}-\\
&\qquad\theta\Big(\{\nabla_{y^0}\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0)\}\{\tilde{\mathbf{a}}^0(t,\mathbf{y}^0)\} + \frac{\partial\tilde{\mathbf{A}}^0}{\partial t}(t,\theta,\mathbf{y}^0)\Big).
\end{aligned} \tag{1.31}$$

On another hand, if $\mathbf{b}$ is linear and independent of t, $\tilde{\alpha}^2$ is given by

$$\begin{aligned}
&\tilde{\alpha}^2(t,\theta,\mathbf{y}^0,\mathbf{y}^1,\mathbf{y}^2) = \big\{\nabla_z\mathbf{Z}(\theta;\mathbf{y}^0)\big\}^{-1}\Big\{\big\{\nabla_x\mathbf{a}(t,\theta,\mathbf{Z}(\theta;\mathbf{y}^0))\big\}\big\{\nabla_z\mathbf{Z}(\theta;\mathbf{y}^0)\big\}\big\{\mathbf{y}^2 + \theta\tilde{\mathbf{A}}^1(t,\theta;\mathbf{y}^0,\mathbf{y}^1)\big\}+\\
&\frac{1}{2}\big\{\nabla_x^2\mathbf{a}(t,\theta,\mathbf{Z}(\theta;\mathbf{y}^0))\big\}\Big\{\big\{\nabla_z\mathbf{Z}(\theta;\mathbf{y}^0)\big\}\big\{\mathbf{y}^1 + \theta\tilde{\mathbf{A}}^0(t,\theta;\mathbf{y}^0)\big\}\Big\}^2-\\
&\quad\theta\Big(\{\nabla_{y^0}\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{y}^1)\}\{\tilde{\mathbf{a}}^0(t,\mathbf{y}^0)\} + \{\nabla_{y^1}\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{y}^1)\}\{\tilde{\mathbf{a}}^1(t,\mathbf{y}^0,\mathbf{y}^1)\} + \frac{\partial\tilde{\mathbf{A}}^1}{\partial t}(t,\theta,\mathbf{y}^0,\mathbf{y}^1)\Big).
\end{aligned} \tag{1.32}$$

The paper is organised as follow: in the second section we briefly prove the Theorems. Section 3 is then devoted to applications to models describing magnetic confinement fusion and isotope resonant separation experiments. Among those applications, one involves a non uniform strong magnetic field, in this case the computations are led using Maple.

REMARK 1.1 With very little changes we can apply the previous results to the case when $\mathbf{b} \equiv \mathbf{b}(t, \theta, \mathbf{x})$ also depends on θ, as soon as the regularity of $\mathbf{b}$ is enough and the assumption (1.4) is realized.

2 Proof of the Theorems

2.1 Sketch of the proof of Theorem 1.2

The key point to prove the results of this paper is Theorem 1.2, in the same spirit as in Schochet [35]. This result is classical and known as Single Phase Averaging. We do not give a detailed proof of it but only a formal sketch of it using expansion (1.24) of $\mathbf{Y}_\varepsilon$.

First, because of the definition (1.13) of $\tilde{\alpha}^0$ and the assumptions (1.2) - (1.4) we deduce that the function $(t, \mathbf{y}^0) \mapsto \tilde{\alpha}^0(t, \frac{t-s}{\varepsilon}, \mathbf{y}^0)$ is regular enough to ensure that, for any $\varepsilon > 0$, $\mathbf{Y}_\varepsilon(\cdot; \mathbf{x}, s)$ exists, is unique on $[s, s+T]$ and remains in a bounded set of $\mathbb{R}^d$ independent of ε.
Then expanding $\tilde{\alpha}^0(\mathbf{Y}_\varepsilon)$, using $\mathbf{Y}_\varepsilon = \mathbf{Y}^0 + \sum_{j\geq 1} \varepsilon^j(\mathbf{Y}^j + [\theta\tilde{\mathbf{A}}^{j-1}]_\varepsilon)$, we obtain

$$\begin{aligned}
\tilde{\alpha}^0(\mathbf{Y}_\varepsilon) = {} & \tilde{\alpha}^0(\mathbf{Y}^0) + \varepsilon\big\{\nabla_{y^0}\tilde{\alpha}^0(\mathbf{Y}^0)\big\}\big\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\big\} + \varepsilon^2\Bigg(\big\{\nabla_{y^0}\tilde{\alpha}^0(\mathbf{Y}^0)\big\}\big\{\mathbf{Y}^2 + [\theta\tilde{\mathbf{A}}^1]_\varepsilon\big\} + \\
& \frac{1}{2}\big\{\nabla^2_{y^0}\tilde{\alpha}^0(\mathbf{Y}^0)\big\}\big\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\big\}^2\Bigg) + \cdots + \varepsilon^k\Bigg(\big\{\nabla_{y^0}\tilde{\alpha}^0(\mathbf{Y}^0)\big\}\big\{\mathbf{Y}^k + [\theta\tilde{\mathbf{A}}^{k-1}]_\varepsilon\big\} + \\
& \frac{1}{2}\big\{\nabla^2_{y^0}\tilde{\alpha}^0(\mathbf{Y}^0)\big\}\big(\sum_{j=1}^{k-1}\{\mathbf{Y}^j + [\theta\tilde{\mathbf{A}}^{j-1}]_\varepsilon, \mathbf{Y}^{k-j} + [\theta\tilde{\mathbf{A}}^{k-j-1}]_\varepsilon\}\big) + \\
& \qquad \cdots + \frac{1}{k!}\big\{\nabla^k_{y^0}\tilde{\alpha}^0(\mathbf{Y}^0)\big\}\big\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\big\}^k\Bigg) + \dots \, .
\end{aligned} \tag{2.1}$$

Plugging the expansion (1.24) in the dynamical system (1.18) and identifying the terms of the same order, we have

$$\frac{\partial \mathbf{Y}^0}{\partial t} + \frac{\partial[\theta\tilde{\mathbf{A}}^0]}{\partial \theta} = \tilde{\alpha}^0(t, \theta, \mathbf{Y}^0), \tag{2.2}$$

at the order -1, and at the order $k-1$ for $k \geq 1$ we have

$$\frac{\partial \mathbf{Y}^k}{\partial t} + \frac{\partial[\theta\tilde{\mathbf{A}}^k]}{\partial \theta} = \tilde{\alpha}^k(t, \theta, \mathbf{Y}^0, \dots \mathbf{Y}^k). \tag{2.3}$$

Then equation (1.16) for $\mathbf{Y}^0$, definition (1.21) of $\tilde{\mathbf{A}}^k$ and equation (1.27) for $\mathbf{Y}^k$ follow, ending the sketch of the proof of Theorem 1.2. ∎

2.2 Proof of Theorem 1.1

Existence and uniqueness of $\mathbf{X}_\varepsilon$ are consequences of existence and uniqueness of $\mathbf{Y}_\varepsilon$. Then the regularity of $\mathbf{Z}$ and Theorem 1.2 allows to pass to the limit in (1.17) in order to deduce (1.14) and (1.16) and thus prove the Theorem. ∎

2.3 Proof of Theorem 1.3

From Theorem 1.2, we deduce

$$\mathbf{Y}_\varepsilon = \mathbf{Y}^0 + \varepsilon(\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon) + \cdots + \varepsilon^{k-1}(\mathbf{Y}^{k-1} + [\theta\tilde{\mathbf{A}}^{k-2}]_\varepsilon) + \varepsilon^k(\mathbf{Y}^k_\varepsilon + [\theta\tilde{\mathbf{A}}^{k-1}]_\varepsilon), \tag{2.4}$$

and then, from (1.17) we deduce

$$\begin{aligned}\mathbf{X}_\varepsilon = [\mathbf{Z}(\mathbf{Y}_\varepsilon)]_\varepsilon = [\mathbf{Z}(\mathbf{Y}^0)]_\varepsilon + \varepsilon\big\{\nabla_z\mathbf{Z}(\mathbf{Y}_\varepsilon)\big\}_\varepsilon\big\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\big\} + \varepsilon^2\bigg(\big\{\nabla_z\mathbf{Z}(\mathbf{Y}_\varepsilon)\big\}_\varepsilon\big\{\mathbf{Y}^2 + [\theta\tilde{\mathbf{A}}^1]_\varepsilon\big\} + \\ \frac{1}{2}\big\{\nabla_z^2\mathbf{Z}(\mathbf{Y}_\varepsilon)\big\}_\varepsilon\big\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\big\}^2\bigg) + \cdots + \varepsilon^k\bigg(\big\{\nabla_z\mathbf{Z}(\mathbf{Y}_\varepsilon)\big\}_\varepsilon\big\{\mathbf{Y}^k_\varepsilon + [\theta\tilde{\mathbf{A}}^{k-1}]_\varepsilon\big\} + \\ \frac{1}{2}\big\{\nabla_z^2\mathbf{Z}(\mathbf{Y}_\varepsilon)\big\}_\varepsilon\Big(\sum_{j=1}^{k-1}\{\mathbf{Y}^j + [\theta\tilde{\mathbf{A}}^{j-1}]_\varepsilon, \mathbf{Y}^{k-j} + [\theta\tilde{\mathbf{A}}^{k-j-1}]_\varepsilon\}\Big) + \\ \cdots + \frac{1}{k!}\big\{\nabla_z^k\mathbf{Z}(\mathbf{Y}^0)\big\}\{\mathbf{Y}^1 + [\theta\tilde{\mathbf{A}}^0]_\varepsilon\}^k\bigg) + o(\varepsilon^k).\end{aligned} \tag{2.5}$$

Comparing this expansion with

$$\mathbf{X}_\varepsilon = [\mathbf{X}^0]_\varepsilon + \varepsilon[\mathbf{X}^1]_\varepsilon + \cdots + \varepsilon^{k-1}[\mathbf{X}^{k-1}]_\varepsilon + \varepsilon^k\mathbf{X}^k_\varepsilon, \tag{2.6}$$

obtained from (1.28) and making the process $\varepsilon \to 0$ give finally Theorem 1.3. ■

3 Application to the gyrokinetic plasma

The dynamical systems associated with equations (1.7), (1.8) and (1.9) are in the form of (1.5), i.e.:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{X}_\varepsilon \\ \mathbf{V}_\varepsilon\end{pmatrix} = \mathbf{a}(t, \frac{t-s}{\varepsilon}, \mathbf{X}_\varepsilon, \mathbf{V}_\varepsilon) + \frac{1}{\varepsilon}\mathbf{b}(t, \mathbf{X}_\varepsilon, \mathbf{V}_\varepsilon)\,, \quad \begin{pmatrix}\mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s) \\ \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s)\end{pmatrix} = \begin{pmatrix}\mathbf{x} \\ \mathbf{v}\end{pmatrix} \tag{3.1}$$

with variable $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ in place of $\mathbf{x}$ and with ad-hoc fields $\mathbf{a}$ and $\mathbf{b}$. Then we can apply our result saying that the solution $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s), \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ can be expanded as

$$\begin{aligned}\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s) = \mathbf{X}^0(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \varepsilon\mathbf{X}^1(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \varepsilon^2\mathbf{X}^2(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \dots, \\ \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s) = \mathbf{V}^0(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \varepsilon\mathbf{V}^1(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \varepsilon^2\mathbf{V}^2(t,\frac{t-s}{\varepsilon};\mathbf{x},\mathbf{v},s) + \dots,\end{aligned} \tag{3.2}$$

and that this expansion may be justified until any order if the regularity of the fields is enough.

In this section, we first lead the computations in the case of Isotope Resonant Separation Regime with the restriction that $\mathcal{M}$ is constant and $\mathbf{B}(t,\theta,\mathbf{x}) = 0$ until the order 1. Then, with the same restrictions, we treat the case of the Guiding Centre Regime until the order 2 and we give the result for the Finite Larmor Radius Regime until the order 0. The forth example concerns the Guiding Centre Regime with a variable strong magnetic field and a constant $\mathbf{B}$ until the order 1.

From the physical point of view, the last example is relevant to understand the behaviour of a plasma in a tokamak. This example shows that the generic computations made before, coupled with the use of Maple, enable to deduce the result relatively comfortably.

3.1 Isotope Resonant Separation Regime with constant strong magnetic field

In the case of Isotope Resonance Separation Regime, i.e. of equation (1.9), $\mathbf{a}$ and $\mathbf{b}$ are:

$$\mathbf{a}(t,\theta,\mathbf{x},\mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}(t,\theta,\mathbf{x}) + \mathbf{v}\times\mathbf{B}(t,\theta,\mathbf{x}) \end{pmatrix}, \quad \mathbf{b}(t,\mathbf{x},\mathbf{v}) = \mathbf{b}(\mathbf{v}) = \begin{pmatrix} 0 \\ \mathbf{v}\times\mathcal{M} \end{pmatrix}. \tag{3.3}$$

For simplicity, we restrict to the case when $\mathbf{B}(t,\theta,\mathbf{x}) = 0$ and when $\mathcal{M} = \mathbf{e}_1$ is a constant vector, $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ being the frame of $\mathbb{R}^3$. Then $\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})$ and $\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})\}^{-1}$ are given by

$$\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w}) = \begin{pmatrix} \mathbf{z} \\ R(\theta)\mathbf{w} \end{pmatrix}, \quad \big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})\big\}^{-1} = \begin{pmatrix} I & 0 \\ 0 & R(-\theta) \end{pmatrix}, \tag{3.4}$$

where $R(\theta)$ is the matrix of the rotation of angle $-\theta$ around $\mathcal{M}$,

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.5}$$

We have the following result.

THEOREM 3.1 *If $\mathbf{E}(t,\theta,\mathbf{x})$ is $C^1_b(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^3)$ and $2\pi-$periodic in θ, the first term of the expansion (3.2) of the solution $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ to*

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{V}_\varepsilon\,, \quad \frac{d\mathbf{V}_\varepsilon}{dt} = \mathbf{E}(t,\frac{t-s}{\varepsilon},\mathbf{X}_\varepsilon) + \frac{1}{\varepsilon}\mathbf{V}_\varepsilon\times\mathcal{M}\,, \quad \mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,, \quad \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}, \tag{3.6}$$

is given by

$$\mathbf{X}^0(t,\theta;\mathbf{x},\mathbf{v},s) = \mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)\,, \quad \mathbf{V}^0(t,\theta;\mathbf{x},\mathbf{v},s) = R(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \tag{3.7}$$

where $(\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s),\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s))$ is solution to

$$\frac{d\mathbf{Y}^0}{dt} = \mathbf{U}^0_\parallel\,, \quad \frac{d\mathbf{U}^0}{dt} = \frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\mathbf{E}(t,\theta,\mathbf{Y}^0)\,d\theta\,, \quad \mathbf{Y}^0(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,, \quad \mathbf{U}^0(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}. \tag{3.8}$$

Proof. In view of (1.13) and (1.12), here we have

$$\tilde{\alpha}^0(t,\theta,\mathbf{y}^0,\mathbf{u}^0) = \begin{pmatrix} R(\theta)\mathbf{u}^0 \\ R(-\theta)\mathbf{E}(t,\theta,\mathbf{y}^0) \end{pmatrix}, \quad \tilde{\mathbf{a}}^0(t,\mathbf{y}^0,\mathbf{u}^0) = \begin{pmatrix} \mathbf{u}^0_\parallel \\ \frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\mathbf{E}(t,\theta,\mathbf{y}^0)\,d\theta \end{pmatrix}. \tag{3.9}$$

Then the proof of the Theorem is straightforward. ∎

In order to obtain the system satisfied by the second term $(\mathbf{Y}^1,\mathbf{U}^1)$ of the expansion we notice that $\mathcal{R}(\theta) = -R(\frac{\pi}{2}+\theta)+R(\frac{\pi}{2})$ is such that $\int_0^\theta R(\sigma)\,d\sigma = \theta P + \mathcal{R}(\theta)$, with P the matrix of the orthogonal projection onto $\mathcal{M}$. We have

$$\mathcal{R}(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin\theta & 1-\cos\theta \\ 0 & \cos\theta-1 & \sin\theta \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

Then, from (1.21), we get:

$$\theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0,\mathbf{u}^0)=\begin{pmatrix}\mathcal{R}(\theta)\mathbf{u}^0\\ \Big(\displaystyle\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\big(R(-\sigma)\mathbf{E}(t,\sigma,\mathbf{y}^0)\big)\end{pmatrix},\tag{3.11}$$

where we denote $\int_0^\theta f(\sigma)\,d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}f(\sigma)\,d\sigma$ by $\big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\big)(f(\sigma))$. We also get

$$\tilde{\alpha}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)=\begin{pmatrix}R(\theta)\Big(\mathbf{u}^1+\Big(\displaystyle\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\Big(R(-\sigma)\mathbf{E}(t,\sigma,\mathbf{y}^0)\Big)\Big)\\ R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{y}^0)(\mathbf{y}^1+\mathcal{R}(\theta)\mathbf{u}^0)\end{pmatrix}-$$
$$\Big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\Big[\begin{pmatrix}R(\sigma)\Big(\displaystyle\frac{1}{2\pi}\int_0^{2\pi}R(-\varsigma)\mathbf{E}(t,\varsigma,\mathbf{y}^0)\,d\varsigma\Big)\\ R(-\sigma)\nabla_x\mathbf{E}(t,\sigma,\mathbf{y}^0)\mathbf{u}^0_{\parallel}\end{pmatrix}+\begin{pmatrix}0\\ R(-\sigma)\displaystyle\frac{\partial\mathbf{E}}{\partial t}(t,\sigma,\mathbf{y}^0)\end{pmatrix}\Big]\tag{3.12}$$

Then using $\big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\big)\big(R(\sigma)\big)=\mathcal{R}(\theta)$ and $\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(\theta)\,d\theta=R(\frac{\pi}{2})-P$, we obtain

$$\tilde{\mathbf{a}}^1(t,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)=$$
$$\begin{pmatrix}\mathbf{u}^1_{\parallel}+\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\Big(R(\theta-\sigma)\mathbf{E}(t,\sigma,\mathbf{y}^0)\Big)d\theta\\ \Big(\displaystyle\frac{1}{2\pi}\int_0^{2\pi}R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{y}^0)d\theta\Big)\mathbf{y}^1+\Big(\frac{1}{2\pi}\int_0^{2\pi}R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{y}^0)\mathcal{R}(\theta)d\theta\Big)\mathbf{u}^0\end{pmatrix}-$$
$$\begin{pmatrix}(R(\frac{\pi}{2})-P)\Big(\displaystyle\frac{1}{2\pi}\int_0^{2\pi}R(-\varsigma)\mathbf{E}(t,\varsigma,\mathbf{y}^0)\,d\varsigma\Big)\\ \Big(\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\Big(R(-\sigma)\nabla_x\mathbf{E}(t,\sigma,\mathbf{y}^0)\Big)d\theta\Big)\mathbf{u}^0_{\parallel}\end{pmatrix}-$$
$$\begin{pmatrix}0\\ \displaystyle\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\Big(R(-\sigma)\frac{\partial\mathbf{E}}{\partial t}(t,\sigma,\mathbf{y}^0)\Big)d\theta\end{pmatrix}.\tag{3.13}$$

Hence we can state the following Theorem.

THEOREM 3.2 *If $\mathbf{E}(t,\theta,\mathbf{x})$ is $C_b^2(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^3)$ and $2\pi-$periodic in θ, the second term of the expansion (3.2) of the solution $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ to (3.6) is given by*

$$\begin{aligned}&\mathbf{X}^1(t,\theta;\mathbf{x},\mathbf{v},s)=\mathbf{Y}^1(t;\mathbf{x},\mathbf{v},s)+\mathcal{R}(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s),\\ &\mathbf{V}^1(t,\theta;\mathbf{x},\mathbf{v},s)=R(\theta)\mathbf{U}^1(t;\mathbf{x},\mathbf{v},s)+R(\theta)\Big(\int_0^\theta d\sigma-\frac{\theta}{2\pi}\int_0^{2\pi}d\sigma\Big)\big(R(-\sigma)\mathbf{E}(t,\sigma,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\big),\end{aligned}\tag{3.14}$$

where $(\mathbf{Y}^1, \mathbf{U}^1)$ *is solution to*

$$
\begin{aligned}
\frac{d\mathbf{Y}^1}{dt} &= \mathbf{U}^1_{\parallel} + \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(\theta-\sigma)\mathbf{E}(t,\sigma,\mathbf{Y}^0)\Big)d\theta - \\
&\qquad\qquad (R(\frac{\pi}{2}) - P)\big(\frac{1}{2\pi}\int_0^{2\pi} R(-\varsigma)\mathbf{E}(t,\varsigma,\mathbf{Y}^0)\,d\varsigma\big), \\
\frac{d\mathbf{U}^1}{dt} &= \big(\frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{Y}^0)d\theta\big)\mathbf{Y}^1 + \big(\frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\nabla_x\mathbf{E}(t,\theta,\mathbf{Y}^0)\mathcal{R}(\theta)d\theta\big)\mathbf{U}^0 - \\
&\quad \big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(-\sigma)\nabla_x\mathbf{E}(t,\sigma,\mathbf{Y}^0)\Big)d\theta\big)\mathbf{U}^0_{\parallel} - \\
&\qquad\qquad \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{\theta} d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(-\sigma)\frac{\partial\mathbf{E}}{\partial t}(t,\sigma,\mathbf{Y}^0)\Big)d\theta, \\
\mathbf{Y}^1&(s;\mathbf{x},\mathbf{v},s) = 0, \quad \mathbf{U}^1(s;\mathbf{x},\mathbf{v},s) = 0.
\end{aligned}
\tag{3.15}
$$

3.2 Guiding Centre Regime with constant strong magnetic field

In the case of the Guiding Centre Regime, i.e. of equation (1.7), we have

$$
\mathbf{a}(t,\theta,\mathbf{x},\mathbf{v}) = \mathbf{a}(t,\mathbf{x},\mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\mathbf{B}(t,\mathbf{x}) \end{pmatrix}, \text{ and } \mathbf{b}(t,\mathbf{x},\mathbf{v}) = \mathbf{b}(\mathbf{v}) = \begin{pmatrix} 0 \\ \mathcal{N} + \mathbf{v}\times\mathcal{M} \end{pmatrix},
\tag{3.16}
$$

As previously, we make the restriction $\mathbf{B}(t,\theta,\mathbf{x}) = \mathcal{N} = 0$ and $\mathcal{M} = \mathbf{e}_1$. Since this situation is similar to the previous one with the only difference that $\mathbf{E}(t,\mathbf{x})$ does not depend on θ we can directly deduce the following Theorem.

Theorem 3.3 *If* $\mathbf{E}(t,\mathbf{x})$ *is* $C^2_b(\mathbb{R}\times\mathbb{R}^3)$*, the first and second terms of the expansion (3.2) of the solution* $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\ \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ *to*

$$
\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{V}_\varepsilon\,, \quad \frac{d\mathbf{V}_\varepsilon}{dt} = \mathbf{E}(t,\mathbf{X}_\varepsilon) + \frac{1}{\varepsilon}\mathbf{V}_\varepsilon\times\mathcal{M}\,, \quad \mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,, \quad \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}, \tag{3.17}
$$

are given by

$$
\mathbf{X}^0(t,\theta;\mathbf{x},\mathbf{v},s) = \mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)\,, \quad \mathbf{V}^0(t,\theta;\mathbf{x},\mathbf{v},s) = R(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \tag{3.18}
$$

and

$$
\begin{aligned}
\mathbf{X}^1(t,\theta;\mathbf{x},\mathbf{v},s) &= \mathbf{Y}^1(t;\mathbf{x},\mathbf{v},s) + \mathcal{R}(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \\
\mathbf{V}^1(t,\theta;\mathbf{x},\mathbf{v},s) &= R(\theta)\mathbf{U}^1(t;\mathbf{x},\mathbf{v},s) + \mathcal{R}(\theta)\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)),
\end{aligned}
\tag{3.19}
$$

where $(\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s),\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s))$ *is solution to*

$$
\frac{d\mathbf{Y}^0}{dt} = \mathbf{U}^0_{\parallel}\,, \quad \frac{d\mathbf{U}^0}{dt} = \mathbf{E}_{\parallel}(t,\mathbf{Y}^0)\,, \quad \mathbf{Y}^0(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,, \quad \mathbf{U}^0(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}, \tag{3.20}
$$

and where $(\mathbf{Y}^1,\mathbf{U}^1)$ *is solution to*

$$
\begin{aligned}
\frac{d\mathbf{Y}^1}{dt} &= \mathbf{U}^1_{\parallel} + (R(\frac{\pi}{2}) - P)\mathbf{E}(t,\mathbf{Y}^0) = \mathbf{U}^1_{\parallel} + \mathbf{E}(t,\mathbf{Y}^0)\times\mathcal{M}, \\
\frac{d\mathbf{U}^1}{dt} &= P\,\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{Y}^1 + \frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)\big(R(-\frac{\pi}{2}) - P\big)\mathbf{U}^0 + \\
&\quad \frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2}) - P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(I-P)\mathbf{U}^0 - (R(-\frac{\pi}{2}) - P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{U}^0_{\parallel} - \\
&\qquad\qquad (R(-\frac{\pi}{2}) - P)\frac{\partial\mathbf{E}}{\partial t}(t,\mathbf{Y}^0), \\
\mathbf{Y}^1&(s;\mathbf{x},\mathbf{v},s) = 0, \quad \mathbf{U}^1(s;\mathbf{x},\mathbf{v},s) = 0.
\end{aligned}
\tag{3.21}
$$

In the computations leading to this Theorem, we use, among other formula,

$$\Big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\Big)\Big(R(-\sigma)\Big) = -\mathcal{R}(-\theta),\ \ \frac{1}{2\pi}\int_0^{2\pi} -\mathcal{R}(-\theta)\,d\theta = R(-\frac{\pi}{2}) - P,\ \ -R(\theta)\mathcal{R}(-\theta) = \mathcal{R}(\theta), \tag{3.22}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathcal{R}(\theta)d\theta = \frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)\big(R(-\frac{\pi}{2})-P\big) + \\ \frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(I-P).\end{aligned} \tag{3.23}$$

Now we turn to the characterisation of $(\mathbf{X}^2,\mathbf{V}^2)$. For this purpose we notice that here

$$\tilde{\alpha}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1) = \\ \begin{pmatrix} R(\theta)\mathbf{u}^1 + \mathcal{R}(\theta)\mathbf{E}(t,\mathbf{y}^0) \\ R(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathbf{y}^1 + R(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathcal{R}(\theta)\mathbf{u}^0 + \mathcal{R}(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathbf{u}^0_{\|} + \mathcal{R}(-\theta)\dfrac{\partial\mathbf{E}}{\partial t}(t,\mathbf{y}^0)\end{pmatrix}. \tag{3.24}$$

In order to get now $\theta\tilde{\mathbf{A}}^1$ we need first to compute

$$\big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\big)\big(\mathcal{R}(\sigma)\big) = I - R(\theta)\,,\ \ \big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\big)\big(\mathcal{R}(-\sigma)\big) = I - R(-\theta). \tag{3.25}$$

Secondly,

$$\begin{aligned}&\big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\big)\big(R(-\sigma)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathcal{R}(\sigma)\big) = \\ &P\nabla_x\mathbf{E}(t,\mathbf{y}^0)(I-R(\theta)) + (I-P)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\begin{pmatrix}0 & 0 & 0\\ 0 & \dfrac{\sin^2\theta}{2} & \sin\theta - \dfrac{\sin 2\theta}{4}\\ 0 & \dfrac{\sin 2\theta}{4} - \sin\theta & \dfrac{\sin^2\theta}{2}\end{pmatrix} + \\ &\big(R(-\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\begin{pmatrix}0 & 0 & 0\\ 0 & -\dfrac{\sin 2\theta}{4} & 1-\cos\theta-\dfrac{\sin^2\theta}{2}\\ 0 & \dfrac{\sin^2\theta}{2}+\cos\theta - 1 & \dfrac{-\sin 2\theta}{4}\end{pmatrix},\end{aligned} \tag{3.26}$$

which also reads

$$\begin{aligned}&\big(\int_0^\theta d\sigma - \frac{\theta}{2\pi}\int_0^{2\pi} d\sigma\big)\big(R(-\sigma)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathcal{R}(\sigma)\big) = P\nabla_x\mathbf{E}(t,\mathbf{y}^0)(I-R(\theta)) + \\ &\frac{1}{2}\big(-\mathcal{R}(-\theta)+R(\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\big(\mathcal{R}(\theta)+R(\frac{\pi}{2})-P\big) + \frac{1}{2}\big(R(-\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\big(R(\frac{\pi}{2})-P\big).\end{aligned} \tag{3.27}$$

Hence integrating (3.24) we have

$$\theta\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1) = \begin{pmatrix}\mathcal{R}(\theta)\mathbf{u}^1 + (I-R(\theta))\mathbf{E}(t,\mathbf{y}^0)\\ -\mathcal{R}(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathbf{y}^1 - (I-R(-\theta))\Big(\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathbf{u}^0_{\|} + \dfrac{\partial\mathbf{E}}{\partial t}(t,\mathbf{y}^0)\Big)\end{pmatrix} +$$

$$\begin{pmatrix} 0 \\ \Big(P\nabla_x\mathbf{E}(t,\mathbf{y}^0)(I-R(\theta))+\dfrac{1}{2}\big(-\mathcal{R}(-\theta)+R(\dfrac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\big(\mathcal{R}(\theta)+R(\dfrac{\pi}{2})-P\big)\Big)\mathbf{u}^0 \end{pmatrix}+$$

$$\begin{pmatrix} 0 \\ \Big(\dfrac{1}{2}\big(R(-\dfrac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\big(R(\dfrac{\pi}{2})-P\big)\mathbf{u}^0 \end{pmatrix}. \tag{3.28}$$

In order to obtain the expression of $\tilde{\alpha}^2$, we need to compute

$$\big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0)\big\}^{-1}\big\{\nabla_{x,v}\mathbf{a}(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0))\big\}\big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0)\big\}$$
$$\left\{\begin{pmatrix}\mathbf{y}^2\\ \mathbf{u}^2\end{pmatrix}+\theta\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)\right\}=$$

$$\begin{pmatrix} 0 & R(\theta) \\ R(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0) & 0\end{pmatrix}\left(\begin{pmatrix}\mathbf{y}^2\\ \mathbf{u}^2\end{pmatrix}+\theta\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)\right)=$$

$$\begin{pmatrix} R(\theta)\mathbf{u}^2+\mathcal{R}(\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathbf{y}^1-(R(\theta)-I)\Big(\nabla_x\mathbf{E}(t,\mathbf{y}^0)\mathbf{u}^0_{\parallel}+\dfrac{\partial\mathbf{E}}{\partial t}(t,\mathbf{y}^0)\Big)+ \\ \Big(P\nabla_x\mathbf{E}(t,\mathbf{y}^0)(I-R(\theta))+\frac{1}{2}\big(R(\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\big(\mathcal{R}(\theta)+R(\frac{\pi}{2})-P\big)+ \\ \frac{1}{2}\big(R(\theta-\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{u}^0 \\ \\ R(-\theta)\nabla_x\mathbf{E}(t,\mathbf{y}^0)\Big(\mathbf{y}^2+\mathcal{R}(\theta)\mathbf{u}^1+(I-R(\theta))\mathbf{E}(t,\mathbf{y}^0)\Big)\end{pmatrix}, \tag{3.29}$$

$$\big\{\nabla^2_{x,v}\mathbf{a}(t,\theta,\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0))\big\}\left\{\Big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{y}^0,\mathbf{u}^0)\Big\}\Big\{\begin{pmatrix}\mathbf{y}^1\\ \mathbf{u}^1\end{pmatrix}+\theta\tilde{\mathbf{A}}^0(t,\theta,\mathbf{y}^0,\mathbf{u}^0)\Big\}\right\}^2=$$
$$\begin{pmatrix} 0 \\ \Big\{\nabla^2_x\mathbf{E}(t,\mathbf{y}^0)\Big\}^2\Big(\{\mathbf{y}^1,\mathbf{y}^1\}+2\{\mathbf{y}^1,\mathcal{R}(\theta)\mathbf{u}^0\}+\{\mathcal{R}(\theta)\mathbf{u}^0,\mathcal{R}(\theta)\mathbf{u}^0\}\Big)\end{pmatrix}. \tag{3.30}$$

A direct but heavy computation also leads the derivatives $\nabla_{\mathbf{y}^0,\mathbf{u}^0}\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)\,\tilde{\mathbf{a}}^0(t,\mathbf{y}^0,\mathbf{u}^0)$, $\nabla_{\mathbf{y}^1,\mathbf{u}^1}\tilde{\mathbf{A}}^1(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)\,\tilde{\mathbf{a}}^1(t,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)$, and $\dfrac{\partial\tilde{\mathbf{A}}^1}{\partial t}(t,\theta,\mathbf{y}^0,\mathbf{u}^0,\mathbf{y}^1,\mathbf{u}^1)$.

Then in view of (1.32), we get $\tilde{\alpha}^2$, which, integrating with respect to θ leads to $\tilde{\mathbf{a}}^2$ and to the equation satisfied by $(\mathbf{X}^2,\mathbf{V}^2)$. Then we have the following Theorem.

THEOREM 3.4 *If $\mathbf{E}(t,\mathbf{x})$ is $C^3_b(\mathbb{R}\times\mathbb{R}^3)$, the third term of the expansion (3.2) of the solution $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s),\ \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ to (3.17) is given by*

$$\begin{aligned}
&\mathbf{X}^2(t,\theta;\mathbf{x},\mathbf{v},s)=\mathbf{Y}^2(t;\mathbf{x},\mathbf{v},s)+\mathcal{R}(\theta)\mathbf{U}^1+(I-R(\theta))\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)),\\
&\mathbf{V}^2(t,\theta;\mathbf{x},\mathbf{v},s)=R(\theta)\mathbf{U}^2(t;\mathbf{x},\mathbf{v},s)+\mathcal{R}(\theta)\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\mathbf{Y}^1(t;\mathbf{x},\mathbf{v},s)-\\
&\quad(R(\theta)-I)\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)\Big(\mathbf{U}^0_{\parallel}(t;\mathbf{x},\mathbf{v},s)+\frac{\partial\mathbf{E}}{\partial t}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\Big)+\\
&\quad\Big(P\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))(I-R(\theta))+\frac{1}{2}\big(R(\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\big(\mathcal{R}(\theta)+R(\frac{\pi}{2})-P\big)+\\
&\quad\quad\frac{1}{2}\big(R(\theta-\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s))\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s).
\end{aligned} \tag{3.31}$$

Moreover $(\mathbf{Y}^2,\mathbf{U}^2)$ *is solution to*

$$\begin{aligned}\frac{d\mathbf{Y}^2}{dt} &= \mathbf{U}^2_\parallel + (R(\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{Y}^1 + (I-P)\Big(\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{U}^0_\parallel + \frac{\partial\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\Big)+\\ &\Big(P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)(I-P) + \big(R(\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{U}^0-\\ &\frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)\big(I-P\big)\mathbf{U}^0 - \frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(R(\frac{\pi}{2})-P)\mathbf{U}^0,\end{aligned} \tag{3.32}$$

$$\begin{aligned}\frac{d\mathbf{U}^2}{dt} &= P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{Y}^2+\\ &\frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)\big(R(-\frac{\pi}{2})-P\big)\mathbf{U}^1 + \frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(I-P)\mathbf{U}^1+\\ &P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{E}(t,\mathbf{Y}^0) - \Big(P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)P + \frac{1}{2}\mathrm{tr}\big((I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(I-P)+\\ &\frac{1}{2}\mathrm{tr}\big((R(-\frac{\pi}{2})-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big)(R(-\frac{\pi}{2})-P)\Big)\mathbf{E}(t,\mathbf{Y}^0)+\\ &\frac{1}{2}\Big\{\nabla_x^2\mathbf{E}(t,\mathbf{Y}^0)\Big\}^2\Big(\{\mathbf{Y}^1,\mathbf{Y}^1\} + 2\{\mathbf{Y}^1,\mathbf{U}^0_\parallel\} + \{(I-P)\mathbf{U}^0,(I-P)\mathbf{U}^0\}+\\ &\qquad\qquad\{(R(-\frac{\pi}{2})-P)\mathbf{U}^0,(R(-\frac{\pi}{2})-P)\mathbf{U}^0\}\Big)+\\ &(P-R(-\frac{\pi}{2}))\{\nabla_x^2\mathbf{E}(t,\mathbf{Y}^0)\}\{\mathbf{Y}^1,\mathbf{U}^0_\parallel\} + (I-P)\Big(\{\nabla_x^2\mathbf{E}(t,\mathbf{Y}^0)\}\{\mathbf{U}^0_\parallel,\mathbf{U}^0_\parallel\} + \frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\mathbf{U}^0_\parallel\Big)-\\ &P\{\nabla_x^2\mathbf{E}(t,\mathbf{Y}^0)\}\{(I-P)\mathbf{U}^0,\mathbf{U}^0_\parallel\}-\\ &\frac{1}{4}(I-P)\{\nabla_x^2\mathbf{E}(t,\mathbf{Y}^0)\}\{(I-P)\mathbf{U}^0,\mathbf{U}^0_\parallel\} - \frac{3}{4}\big(R(-\frac{\pi}{2})-P\big)\{\nabla_x^2\mathbf{E}(t,\mathbf{Y}^0)\}\{\big(R(\frac{\pi}{2})-P\big)\mathbf{U}^0,\mathbf{U}^0_\parallel\}-\\ &\Big(-(I-P)\Big(\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\Big) + P\nabla_x\mathbf{E}(t,\mathbf{Y}^0)(I-P)+\\ &\qquad\frac{1}{4}(I-P)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)(I-P) + \frac{3}{4}\big(R(-\frac{\pi}{2})-P\big)\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{E}_\parallel(t,\mathbf{Y}^0)+\\ &(P-R(-\frac{\pi}{2}))\nabla_x\mathbf{E}(t,\mathbf{Y}^0)\mathbf{U}^1_\parallel + (P-R(-\frac{\pi}{2}))\nabla_x\mathbf{E}(t,\mathbf{y}^0)(R(\frac{\pi}{2})-P)\mathbf{E}(t,\mathbf{Y}^0)+\\ &(P-R(-\frac{\pi}{2}))\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\mathbf{Y}^1 - (I-P)\Big(\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\mathbf{U}^0_\parallel + \frac{\partial^2\mathbf{E}}{\partial t^2}(t,\mathbf{Y}^0)\Big) - P\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)(I-P)-\\ &\Big(\frac{1}{4}(I-P)\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)(I-P) + \frac{3}{4}\big(R(-\frac{\pi}{2})-P\big)\frac{\partial\nabla_x\mathbf{E}}{\partial t}(t,\mathbf{Y}^0)\big(R(\frac{\pi}{2})-P\big)\Big)\mathbf{U}^0.\end{aligned} \tag{3.33}$$

3.3 Finite Larmor Radius Regime with constant strong magnetic field

In the case of Finite Larmor Radius Regime (1.8) with $\mathcal{M}=\mathbf{e}_1$, we have

$$\mathbf{a}(t,\theta,\mathbf{x},\mathbf{v}) = \begin{pmatrix}\mathbf{v}_\parallel\\ \mathbf{E}(t,\mathbf{x})\end{pmatrix} \text{ and } \mathbf{b}(t,\mathbf{x},\mathbf{v}) = \mathbf{b}(\mathbf{v}) = \begin{pmatrix}\mathbf{v}_\perp\\ \mathbf{v}\times\mathcal{M}\end{pmatrix}, \tag{3.34}$$

$$\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w}) = \begin{pmatrix}\mathbf{z}+\mathcal{R}(\theta)\mathbf{w}\\ R(\theta)\mathbf{w}\end{pmatrix},\quad \big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{x},\mathbf{v})\big\}^{-1} = \begin{pmatrix}I & \mathcal{R}(-\theta)\\ 0 & R(-\theta)\end{pmatrix}. \tag{3.35}$$

with R and $\mathcal{R}$ defined by (3.5) and (3.10).

Here we give the result for this case only for the order 0. We have

$$\tilde{\alpha}^0(t,\theta,\mathbf{y}^0,\mathbf{u}^0) = \begin{pmatrix}\mathbf{u}^0_\parallel + \mathcal{R}(-\theta)\mathbf{E}(\mathbf{y}^0+\mathcal{R}(\theta)\mathbf{u}^0,t)\\ R(-\theta)\mathbf{E}(\mathbf{y}^0+\mathcal{R}(\theta)\mathbf{u}^0,t)\end{pmatrix}, \tag{3.36}$$

and

$$\tilde{\mathbf{a}}^0(t,\mathbf{y}^0,\mathbf{u}^0) = \begin{pmatrix} \mathbf{u}^0_{\|} + \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \mathcal{R}(-\theta)\mathbf{E}(\mathbf{y}^0 + \mathcal{R}(\theta)\mathbf{u}^0, t)\, d\theta \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} R(-\theta)\mathbf{E}(\mathbf{y}^0 + \mathcal{R}(\theta)\mathbf{u}^0, t)\, d\theta \end{pmatrix}. \tag{3.37}$$

Hence we have the following Theorem.

THEOREM 3.5 *If we assume that* $\mathbf{E}(t,\mathbf{x})$ *is* $C^1_b(\mathbb{R}\times\mathbb{R}^3)$*, the first term of the expansion (3.2) of the solution* $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s), \mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ *to*

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{V}_{\varepsilon\|} + \frac{1}{\varepsilon}\mathbf{V}_{\varepsilon\perp}\,, \quad \frac{d\mathbf{V}_\varepsilon}{dt} = \mathbf{E}(t,\mathbf{X}_\varepsilon) + \frac{1}{\varepsilon}\mathbf{V}_\varepsilon\times\mathcal{M}\,,\ \mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,,\ \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}, \tag{3.38}$$

is given by

$$\mathbf{X}^0(t,\theta;\mathbf{x},\mathbf{v},s) = \mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s) + \mathcal{R}(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s)\,,\quad \mathbf{V}^0(t,\theta;\mathbf{x},\mathbf{v},s) = R(\theta)\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \tag{3.39}$$

where $(\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s), \mathbf{U}^0(t;\mathbf{x},\mathbf{v},s))$ *is solution to*

$$\frac{d\mathbf{Y}^0}{dt} = \mathbf{U}^0_{\|} + \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}(-\theta)\mathbf{E}(\mathbf{Y}^0 + \mathcal{R}(\theta)\mathbf{U}^0, t)\, d\theta\,,\ \frac{d\mathbf{U}^0}{dt} = \frac{1}{2\pi}\int_0^{2\pi} R(-\theta)\mathbf{E}(\mathbf{Y}^0 + \mathcal{R}(\theta)\mathbf{U}^0, t)\, d\theta, \tag{3.40}$$

with the initial conditions

$$\mathbf{Y}^0(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,,\ \ \mathbf{U}^0(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}. \tag{3.41}$$

3.4 Guiding Centre Regime with variable strong magnetic field

Here we study a situation representative of what happens inside a tokamak, i.e., the Guiding Centre Regime with a variable $\mathcal{M}$, with $\mathcal{N} = 0$ and $\mathbf{B} = \mathbf{e}_3$. In other words we consider the following system:

$$\frac{d\mathbf{X}_\varepsilon}{dt} = \mathbf{V}_\varepsilon\,,\ \frac{d\mathbf{V}_\varepsilon}{dt} = \mathbf{E}(t,\mathbf{X}_\varepsilon) + \mathbf{V}_\varepsilon\times\mathbf{e}_3 + \frac{1}{\varepsilon}\mathbf{V}_\varepsilon\times\mathcal{M}(\mathbf{X}_\varepsilon)\,,\ \mathbf{X}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{x}\,,\ \mathbf{V}_\varepsilon(s;\mathbf{x},\mathbf{v},s) = \mathbf{v}, \tag{3.42}$$

where

$$\mathcal{M}(\mathbf{x}) = \frac{1}{\sqrt{{x_1}^2 + {x_2}^2}}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = \rho^T(\mathbf{x})\mathbf{e}_1,\ \text{with } \rho(\mathbf{x}) = \frac{1}{\sqrt{{x_1}^2 + {x_2}^2}}\begin{pmatrix} -x_2 & x_1 & 0 \\ -x_1 & x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.43}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ in the frame $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ of $\mathbb{R}^3$. In this case

$$\mathbf{a}(t,\theta,\mathbf{x},\mathbf{v}) = \mathbf{a}(t,\mathbf{x},\mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\mathbf{e}_3 \end{pmatrix},\ \text{and } \mathbf{b}(t,\mathbf{x},\mathbf{v}) = \mathbf{b}(\mathbf{x},\mathbf{v}) = \begin{pmatrix} 0 \\ \mathbf{v}\times\mathcal{M}(\mathbf{x}) \end{pmatrix}, \tag{3.44}$$

and thus, $R(\theta)$ being defined by (3.5),

$$\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w}) = \begin{pmatrix} \mathbf{z} \\ \rho^T(\mathbf{z})R(\theta)\rho(\mathbf{z})\mathbf{w} \end{pmatrix},\ \text{where } \rho^T R\rho = \begin{pmatrix} \frac{{z_2}^2 + {z_1}^2\cos(\theta)}{{z_1}^2+{z_2}^2} & \frac{z_1 z_2(\cos(\theta)-1)}{{z_1}^2+{z_2}^2} & -\frac{z_1\sin(\theta)}{\sqrt{{z_1}^2+{z_2}^2}} \\ \frac{z_1 z_2(\cos(\theta)-1)}{{z_1}^2+{z_2}^2} & \frac{{z_1}^2 + {z_2}^2\cos(\theta)}{{z_1}^2+{z_2}^2} & -\frac{z_2\sin(\theta)}{\sqrt{{z_1}^2+{z_2}^2}} \\ \frac{z_1\sin(\theta)}{\sqrt{{z_1}^2+{z_2}^2}} & \frac{z_2\sin(\theta)}{\sqrt{{z_1}^2+{z_2}^2}} & \cos(\theta) \end{pmatrix}, \tag{3.45}$$

and we have the following Theorem.

Theorem 3.6 *If* $\mathbf{E}(t,\mathbf{x})$ *is* $C_b^2(\mathbb{R}\times\mathbb{R}^3)$, *the first term of the expansion (3.2) of the solution* $(\mathbf{X}_\varepsilon(t;\mathbf{x},\mathbf{v},s)$, $\mathbf{V}_\varepsilon(t;\mathbf{x},\mathbf{v},s))$ *to (3.42) is given by*

$$\mathbf{X}^0(t,\theta;\mathbf{x},\mathbf{v},s)=\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s)\,,\quad \mathbf{V}^0(t,\theta;\mathbf{x},\mathbf{v},s)=\rho^T(\mathbf{x})R(\theta)\rho(\mathbf{x})\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s), \tag{3.46}$$

where $(\mathbf{Y}^0(t;\mathbf{x},\mathbf{v},s),\mathbf{U}^0(t;\mathbf{x},\mathbf{v},s))$ *is solution to*

$$\begin{gathered}\frac{d\mathbf{Y}^0}{dt}=\overline{A}(\mathbf{Y}^0)\mathbf{U}^0,\qquad \frac{d\mathbf{U}^0}{dt}=\overline{\beta}(\mathbf{Y}^0,\mathbf{U}^0)+\overline{A}(\mathbf{Y}^0)\mathbf{E}_{\|}(t,\mathbf{Y}^0)+\mathbf{U}^0\times\overline{A}(\mathbf{Y}^0)\mathbf{e}_3,\\
\text{with } \overline{A}(\mathbf{y})=\frac{1}{{y_1}^2+{y_2}^2}\begin{pmatrix}{y_2}^2 & -y_1y_2 & 0\\ -y_1y_2 & {y_1}^2 & 0\\ 0 & 0 & 0\end{pmatrix},\ \text{and } \overline{\beta}(\mathbf{y},\mathbf{u})=\begin{pmatrix}\dfrac{(y_2u_1-y_1u_2)u_2}{{y_1}^2+{y_2}^2}\\ \dfrac{(y_1u_2-y_2u_1)u_1}{{y_1}^2+{y_2}^2}\\ 0\end{pmatrix},\\
\mathbf{Y}^0(s;\mathbf{x},\mathbf{v},s)=\mathbf{x},\ ,\quad \mathbf{U}^0(s;\mathbf{x},\mathbf{v},s)=\mathbf{v}.\end{gathered} \tag{3.47}$$

The term $\mathbf{X}^1(t,\theta;\mathbf{x},\mathbf{v},s)$ *is given by*

$$\begin{aligned}
\mathbf{X}^1{}_1(t,\theta;\mathbf{x},\mathbf{v},s)&=\frac{1}{\Omega^2}\Big(\mathbf{Y}^0{}_1\Omega(\cos(\theta)-1)\mathbf{U}^0{}_3+\mathbf{Y}^0{}_1(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\sin(\theta)+\mathbf{Y}^1{}_1\Omega^2\Big),\\
\mathbf{X}^1{}_2(t,\theta;\mathbf{x},\mathbf{v},s)&=\frac{1}{\Omega^2}\Big(\mathbf{Y}^0{}_2\Omega(\cos(\theta)-1)\mathbf{U}^0{}_3+\mathbf{Y}^0{}_2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\sin(\theta)+\mathbf{Y}^1{}_2\Omega^2\Big),\\
\mathbf{X}^1{}_3(t,\theta;\mathbf{x},\mathbf{v},s)&=\frac{1}{\Omega}\Big((-\mathbf{Y}^0{}_1\mathbf{U}^0{}_1-\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)(\cos(\theta)-1)+\mathbf{Y}^1{}_3\Omega+\sin(\theta)\mathbf{U}^0{}_3\Omega\Big),
\end{aligned} \tag{3.48}$$

where $\Omega=\sqrt{{\mathbf{Y}^0{}_1}^2+{\mathbf{Y}^0{}_2}^2}$ *and where* $\mathbf{Y}^1(t;\mathbf{x},\mathbf{v},s)$ *is solution to*

$$\begin{aligned}
\frac{d\mathbf{Y}^1{}_1}{dt}=&\bigg(\Big(-\Omega^2\mathbf{Y}^0{}_2\mathbf{U}^0{}_2+2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{U}^0{}_2\mathbf{Y}^0{}_1)\Big)\mathbf{Y}^1{}_1+\Big((2\mathbf{U}^0{}_1\mathbf{Y}^0{}_2-\mathbf{U}^0{}_2\mathbf{Y}^0{}_1)\Omega^2+\\
&2{\mathbf{Y}^0{}_2}^2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{U}^0{}_2\mathbf{Y}^0{}_1)\Big)\mathbf{Y}^1{}_2+{\mathbf{Y}^0{}_2}^2\mathbf{U}^1{}_1\Omega^2-\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_2-\Omega^3\mathbf{Y}^0{}_1\mathbf{E}_3-\mathbf{Y}^0{}_2\mathbf{U}^0{}_3\Omega^3-\\
&2\mathbf{Y}^0{}_2\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{U}^0{}_2\mathbf{Y}^0{}_1)\Omega\bigg)\Big/\Omega^4,\\[2ex]
\frac{d\mathbf{Y}^1{}_2}{dt}=&\bigg(\Big((2\mathbf{U}^0{}_2\mathbf{Y}^0{}_1-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2)\Omega^2-2{\mathbf{Y}^0{}_1}^2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{U}^0{}_2\mathbf{Y}^0{}_1)\Big)\mathbf{Y}^1{}_1+\Big(-\Omega^2\mathbf{Y}^0{}_1\mathbf{U}^0{}_1-\\
&2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{U}^0{}_2\mathbf{Y}^0{}_1)\Big)\mathbf{Y}^1{}_2-\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_1+{\mathbf{Y}^0{}_1}^2\mathbf{U}^1{}_2\Omega^2-\\
&\Omega^3\mathbf{Y}^0{}_2\mathbf{E}_3+\Omega^3\mathbf{Y}^0{}_1\mathbf{U}^0{}_3+2\mathbf{Y}^0{}_1\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+\mathbf{U}^0{}_2\mathbf{Y}^0{}_1)\Omega\bigg)\Big/\Omega^4,\\[2ex]
\frac{d\mathbf{Y}^1{}_3}{dt}=&\bigg(\Omega^2\mathbf{Y}^0{}_1\mathbf{E}_1+\Omega^2\mathbf{Y}^0{}_2\mathbf{E}_2+\Big(\mathbf{U}^0{}_2\mathbf{Y}^0{}_1+{\mathbf{U}^0{}_1}^2-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2+{\mathbf{U}^0{}_2}^2\Big)\Omega^2-\\
&\Big(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1+\mathbf{Y}^0{}_2\mathbf{U}^0{}_2\Big)^2\bigg)\Big/\Omega^3,\\
\mathbf{Y}^1(s;\mathbf{x},\mathbf{v},s)=&\,0,
\end{aligned} \tag{3.49}$$

with $\mathbf{E}$ *being evaluated in* $\mathbf{Y}^0$. *The result concerning* $\mathbf{V}^1(t,\theta;\mathbf{x},\mathbf{v},s)$ *is given in the appendix A.*

This Theorem is the consequence of the following computations. Setting $A=\rho^TR\rho$ we have

$$\big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})\big\}=\begin{pmatrix}I & 0\\ \nabla_z(A\mathbf{w}) & A\end{pmatrix}\ \text{and}\ \big\{\nabla_{z,w}\mathbf{Z}(t,\theta;\mathbf{z},\mathbf{w})\big\}^{-1}=\begin{pmatrix}I & 0\\ A^T\nabla_z(A\mathbf{w}) & A^T\end{pmatrix}. \tag{3.50}$$

Then, since in this example $\frac{\partial \mathbf{Z}}{\partial t} = 0$, and

$$\mathbf{a}(t, \mathbf{Z}(t, \theta; \mathbf{z}, \mathbf{w})) = \begin{pmatrix} A\mathbf{w} \\ \mathbf{E} + A\mathbf{w} \times \mathbf{e}_3 \end{pmatrix}, \tag{3.51}$$

applying (1.12) and (1.13) we get

$$\tilde{\alpha}^0(t, \theta, \mathbf{z}, \mathbf{w}) = \begin{pmatrix} A\mathbf{w} \\ -A^T \nabla_z (A\mathbf{w})(A\mathbf{w}) + A^T \mathbf{E} + \mathbf{w} \times A^T \mathbf{e}_3 \end{pmatrix}, \tag{3.52}$$

and since $\int_0^{2\pi} A d\theta = \int_0^{2\pi} A^T d\theta = 2\pi \overline{A}$, and $\int_0^{2\pi} -A^T \nabla_z (A\mathbf{w})(A\mathbf{w}) = \overline{\beta}$, we finally get the first part of the Theorem.

The second part of the Theorem is obtained following the computation program described in the previous sections using Maple.

A Appendix : $\mathbf{V}^1$ for the Guiding Centre Regime with variable strong magnetic field

The computations of this appendix have been realized using Maple5.5, on computers of Medicis Centre, Ecole Polytechnique, Palaiseau, France.

The electric field $\mathbf{E}$ and its derivatives being evaluated in $\mathbf{Y}^0$, and Ω standing for $\sqrt{(\mathbf{Y}^0{}_1)^2 + (\mathbf{Y}^0{}_2)^2}$, the term $\mathbf{V}^1(t, \theta; \mathbf{x}, \mathbf{v}, s)$ of the expansion (3.2) of the solution to (3.42) is given by :

$$\begin{multline}
\mathbf{V}^1{}_1(t; \mathbf{x}, \mathbf{v}, s) = \Bigg(\Bigg(\Big((2\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Omega^2 - 2\mathbf{Y}^0{}_1{}^2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_1 + \\
\Big(\mathbf{Y}^0{}_1\mathbf{U}^0{}_2\Omega^2 - 2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_2 + \mathbf{Y}^0{}_1{}^2\mathbf{U}^1{}_1\Omega^2 + \mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_2 + \Omega^3\mathbf{Y}^0{}_1\mathbf{E}_3 + \\
\Omega^3\mathbf{Y}^0{}_2\mathbf{U}^0{}_3 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega\Bigg)\cos(\theta) + \Big((-\Omega^3\mathbf{U}^0{}_3 + \Omega\mathbf{U}^0{}_3\mathbf{Y}^0{}_1{}^2)\mathbf{Y}^1{}_1 + \\
\mathbf{Y}^0{}_1\mathbf{U}^0{}_3\mathbf{Y}^0{}_2\Omega\mathbf{Y}^1{}_2 - \mathbf{Y}^0{}_1\Omega^3\mathbf{U}^1{}_3 + \Omega^2\mathbf{Y}^0{}_1{}^2\mathbf{E}_1 + \Omega^2\mathbf{Y}^0{}_1\mathbf{E}_2\mathbf{Y}^0{}_2 + \mathbf{U}^0{}_2\Omega^4 + \\
\mathbf{U}^0{}_2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega^2\Big)\sin(\theta) + \Big(-\Omega^2\mathbf{Y}^0{}_2\mathbf{U}^0{}_2 + 2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_1 + \\
\Big((2\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 - \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega^2 + 2\mathbf{Y}^0{}_2{}^2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_2 + \mathbf{Y}^0{}_2{}^2\mathbf{U}^1{}_1\Omega^2 - \mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_2 - \\
\Omega^3\mathbf{Y}^0{}_1\mathbf{E}_3 - \Omega^3\mathbf{Y}^0{}_2\mathbf{U}^0{}_3 - \mathbf{Y}^0{}_2\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega\Bigg)\Bigg/\Omega^4, \tag{A.1}
\end{multline}$$

$$\begin{multline}
\mathbf{V}^1{}_2(t; \mathbf{x}, \mathbf{v}, s) = \Bigg(\Bigg((\mathbf{U}^0{}_1\mathbf{Y}^0{}_2\Omega^2 - 2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_2))\mathbf{Y}^1{}_1 + \Big((\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + 2\mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Omega^2 - \\
2\mathbf{Y}^0{}_2{}^2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_2 + \mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_1 + \mathbf{Y}^0{}_2{}^2\mathbf{U}^1{}_2\Omega^2 + \Omega^3\mathbf{Y}^0{}_2\mathbf{E}_3 - \Omega^3\mathbf{Y}^0{}_1\mathbf{U}^0{}_3 - \\
\mathbf{Y}^0{}_1\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega\Bigg)\cos(\theta) + \Big(\mathbf{Y}^0{}_1\mathbf{U}^0{}_3\mathbf{Y}^0{}_2\Omega\mathbf{Y}^1{}_1 + (-\Omega^3\mathbf{U}^0{}_3 + \Omega\mathbf{U}^0{}_3\mathbf{Y}^0{}_2{}^2)\mathbf{Y}^1{}_2 - \\
\mathbf{Y}^0{}_2\Omega^3\mathbf{U}^1{}_3 + \Omega^2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\mathbf{E}_1 + \Omega^2\mathbf{Y}^0{}_2{}^2\mathbf{E}_2 - \mathbf{U}^0{}_1\Omega^4 - \mathbf{U}^0{}_1(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega^2\Big)\sin(\theta) + \\
\Big((2\mathbf{Y}^0{}_1\mathbf{U}^0{}_2 - \mathbf{U}^0{}_1\mathbf{Y}^0{}_2)\Omega^2 - 2\mathbf{Y}^0{}_1{}^2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Big)\mathbf{Y}^1{}_1 + \\
(-\Omega^2\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 - 2\mathbf{Y}^0{}_1\mathbf{Y}^0{}_2(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2))\mathbf{Y}^1{}_2 - \mathbf{Y}^0{}_1\mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_1 + \mathbf{Y}^0{}_1{}^2\mathbf{U}^1{}_2\Omega^2 - \\
\Omega^3\mathbf{Y}^0{}_2\mathbf{E}_3 + \Omega^3\mathbf{Y}^0{}_1\mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_3(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega\Bigg)\Bigg/\Omega^4, \tag{A.2}
\end{multline}$$

$$
\begin{aligned}
\mathbf{V}^1{}_3(t;\mathbf{x},\mathbf{v},s) = \Bigg(\bigg(&\Omega^3\mathbf{U}^1{}_3 - \Omega^2\mathbf{Y}^0{}_1\mathbf{E}_1 - \Omega^2\mathbf{Y}^0{}_2\mathbf{E}_2 + (\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 - \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)\Omega^2 - \\
&(-\mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2)^2\bigg)\cos(\theta) + \bigg((\mathbf{U}^0{}_1\Omega^2 - \mathbf{Y}^0{}_1(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_2))\mathbf{Y}^1{}_1 + \\
&(\mathbf{U}^0{}_2\Omega^2 - \mathbf{Y}^0{}_2(\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_2))\mathbf{Y}^1{}_2 + \mathbf{Y}^0{}_2\Omega^2\mathbf{U}^1{}_2 + \Omega^3\mathbf{E}_3 + \mathbf{Y}^0{}_1\Omega^2\mathbf{U}^1{}_1\bigg)\sin(\theta) + \\
&\Omega^2\mathbf{Y}^0{}_1\mathbf{E}_1 + \Omega^2\mathbf{Y}^0{}_2\mathbf{E}_2 + ({\mathbf{U}^0{}_1}^2 + \mathbf{Y}^0{}_1\mathbf{U}^0{}_2 - \mathbf{U}^0{}_1\mathbf{Y}^0{}_2 + {\mathbf{U}^0{}_2}^2)\Omega^2 - (\mathbf{Y}^0{}_1\mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\mathbf{U}^0{}_2)^2\Bigg)\Big/\Omega^3, \quad \text{(A.3)}
\end{aligned}
$$

where $\mathbf{U}^1(t;\mathbf{x},\mathbf{v},s))$ is solution to:

$$
\begin{aligned}
&\frac{d\mathbf{U}^1{}_1}{dt} = 1/4 \Bigg(\bigg(-4\, \mathbf{Y}^0{}_2\, \Omega^8\, \mathbf{Y}^0{}_1\, \frac{\partial \mathbf{E}_2}{\partial x_3} + 4{\mathbf{Y}^0{}_2}^2\, \Omega^8\, \frac{\partial \mathbf{E}_1}{\partial x_3} \bigg)\, \mathbf{Y}^1{}_3 + \bigg(-4\Omega^9\, \mathbf{Y}^0{}_2 + (-{\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3 + 4\mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2\, \mathbf{Y}^0{}_1 - 4\mathbf{U}^0{}_1 \\
&{\mathbf{Y}^0{}_2}^2)\, \Omega^7 + (-{\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_1 - 2\mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1 + 2\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_1 + 2{\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 + {\mathbf{Y}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3 - \mathbf{U}^0{}_2\, \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^3 - 2\, {\mathbf{Y}^0{}_1}^2 \\
&\mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^2 + {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3\, \mathbf{Y}^0{}_2 - 4\, \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_2}^3\, \mathbf{Y}^0{}_1)\, \Omega^5 + \mathbf{Y}^0{}_1\, (3\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2 + {\mathbf{Y}^0{}_2}^4 + {\mathbf{Y}^0{}_1}^4 + \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^3)\, (\mathbf{Y}^0{}_1\, \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2 \\
&\mathbf{U}^0{}_2)\, \Omega^3\bigg)\, \mathbf{E}_3 + \bigg(4\, \Omega^8\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2 - {\mathbf{Y}^0{}_1}^3\, \Omega^6\, \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^2\, (-{\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 - \mathbf{U}^0{}_1\, \mathbf{Y}^0{}_2\, \mathbf{Y}^0{}_1 + {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3 + \mathbf{U}^0{}_3\, \mathbf{Y}^0{}_1\, \mathbf{Y}^0{}_2) \\
&\Omega^4 + {\mathbf{Y}^0{}_1}^4\, \mathbf{Y}^0{}_2\, (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\, (\mathbf{Y}^0{}_1\, \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2)\, \Omega^2\bigg)\, \mathbf{U}^1{}_1 + \bigg(-\Omega^7\, \mathbf{Y}^0{}_1\, \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1\, (8\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2\, \mathbf{Y}^0{}_1 - \mathbf{U}^0{}_1 \\
&\mathbf{Y}^0{}_2\, \mathbf{Y}^0{}_1 + 5\, \mathbf{U}^0{}_3\, \mathbf{Y}^0{}_1\, \mathbf{Y}^0{}_2 + 2\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3)\, \Omega^5 - {\mathbf{Y}^0{}_1}^2\, (8\, \mathbf{Y}^0{}_1\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^2 - \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_1 + 7\, {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2 \\
&\mathbf{U}^0{}_2 + 6\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3 + 7\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3 - {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_1 - {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_2)\, \Omega^3\bigg)\, {\mathbf{Y}^1{}_1}^2 + \bigg({\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^5\, {\mathbf{U}^0{}_2}^3 - 8\, {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^7 + 7 \\
&{\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^7 + {\mathbf{Y}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^5\, {\mathbf{U}^0{}_1}^3 + 6\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^7\, {\mathbf{U}^0{}_2}^2 + {\mathbf{Y}^0{}_1}^6\, {\mathbf{U}^0{}_1}^3\, \mathbf{Y}^0{}_2 + {\mathbf{Y}^0{}_2}^4\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{U}^0{}_2}^3 + 5\, {\mathbf{U}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^5\, {\mathbf{Y}^0{}_2}^2 \\
&\mathbf{U}^0{}_3 + \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_1}^7 - 5\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_2}^2 + 3\, {\mathbf{Y}^0{}_1}^5\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2\, {\mathbf{U}^0{}_1}^2 + 19\, {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_2}^2\, {\mathbf{Y}^0{}_2}^3 + 9\, {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3 \\
&{\mathbf{Y}^0{}_1}^6\, \mathbf{Y}^0{}_2 - 9\, {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^6\, \mathbf{Y}^0{}_1 + {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^4\, {\mathbf{Y}^0{}_2}^3 - 16\, {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^4\, {\mathbf{Y}^0{}_1}^3 + 3\, {\mathbf{Y}^0{}_2}^4\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{U}^0{}_2}^2\, \mathbf{U}^0{}_1 + {\mathbf{Y}^0{}_1}^5 \\
&{\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_1}^2 + 3\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_1}^4\, {\mathbf{U}^0{}_2}^2 - 15\, {\mathbf{Y}^0{}_2}^5\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3 + 3\, {\mathbf{Y}^0{}_2}^3\, {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_2\, {\mathbf{U}^0{}_1}^2 - 5\, {\mathbf{Y}^0{}_2}^6\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_2}^2 \\
&\mathbf{Y}^0{}_1 + 10\, {\mathbf{Y}^0{}_2}^5\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^2 + 8\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^6 - 6\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^5\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1 + 19\, {\mathbf{Y}^0{}_1}^5\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1 + 7 \\
&{\mathbf{Y}^0{}_1}^6\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1 + 3\, {\mathbf{Y}^0{}_1}^4\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_2 + 16\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_2\bigg)\, \Omega + (-4\, \Omega^8\, \mathbf{U}^0{}_2\, \mathbf{Y}^0{}_1 - {\mathbf{Y}^0{}_1}^2\, \Omega^6 \\
&\mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1\, \bigg(\mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3 - \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_2}^4 + \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_2}^3\, \mathbf{Y}^0{}_1 - \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1 + {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^2 + {\mathbf{Y}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3 + \mathbf{U}^0{}_2 \\
&\mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^3 + {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_1\bigg)\, \Omega^4 - {\mathbf{Y}^0{}_1}^5\, (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\, (\mathbf{Y}^0{}_1\, \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2)\, \Omega^2)\, \mathbf{U}^1{}_2 + \bigg(6\, \Omega^6\, {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{E}_1 - 2\, \mathbf{Y}^0{}_1\, (-{\mathbf{Y}^0{}_2}^2 + 2 \\
&{\mathbf{Y}^0{}_1}^2)\, \Omega^6\, \mathbf{E}_2 + 4\, \Omega^8\, {\mathbf{Y}^0{}_2}^2\, \frac{\partial \mathbf{E}_1}{\partial x_2} - 4\, \Omega^8\, \mathbf{Y}^0{}_1\, \frac{\partial \mathbf{E}_2}{\partial x_2}\, \mathbf{Y}^0{}_2 - \Omega^8\, \mathbf{Y}^0{}_1\, \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_1\, (-\mathbf{Y}^0{}_1\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2 + 3\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_3}^2 + \mathbf{U}^0{}_1\, \mathbf{Y}^0{}_1 \\
&\mathbf{U}^0{}_2 + 12\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_2}^2 + 2\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2\, \mathbf{Y}^0{}_1 + 4\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_1}^2 + 3\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^2 + \mathbf{Y}^0{}_1\, {\mathbf{U}^0{}_3}^2)\, \Omega^6 + (-{\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{U}^0{}_2}^2 + 2 \\
&{\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2 + 10\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_2 + \mathbf{U}^0{}_3\, \mathbf{U}^0{}_1\, \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^3 - 3\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, {\mathbf{U}^0{}_2}^2 - 8\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_3}^2\, {\mathbf{Y}^0{}_1}^3 - {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_2 \\
&\mathbf{U}^0{}_1 - 3\, {\mathbf{U}^0{}_3}^2\, {\mathbf{Y}^0{}_1}^4 + \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_1}^3\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3 + 2\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{U}^0{}_1}^2 - 9\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{U}^0{}_3}^2\, {\mathbf{Y}^0{}_1}^2 - 2\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_1\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2 + 2\, {\mathbf{Y}^0{}_1}^3 \\
&\mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_1}^2 - 4\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_2 - 6\, {\mathbf{U}^0{}_1}^2\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3 - 8\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^4 - {\mathbf{Y}^0{}_2}^3\, {\mathbf{U}^0{}_3}^2\, \mathbf{Y}^0{}_1)\, \Omega^4 - \mathbf{Y}^0{}_2\, (-8\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^5 - 2\, \mathbf{Y}^0{}_1 \\
&{\mathbf{Y}^0{}_2}^4\, {\mathbf{U}^0{}_2}^2 - 7\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_3}^2\, {\mathbf{Y}^0{}_1}^4 - {\mathbf{Y}^0{}_2}^4\, {\mathbf{U}^0{}_3}^2\, \mathbf{Y}^0{}_1 + 9\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^3\, {\mathbf{U}^0{}_2}^2 - 2\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2 - 4\, {\mathbf{Y}^0{}_1}^4\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_1}^2 - 9 \\
&{\mathbf{Y}^0{}_1}^4\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1 - 14\, {\mathbf{Y}^0{}_2}^3\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{U}^0{}_1}^2 - 9\, {\mathbf{U}^0{}_3}^2\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^2 - 7\, {\mathbf{Y}^0{}_1}^5\, {\mathbf{U}^0{}_3}^2 + 11\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{U}^0{}_2}^2 + \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^4 \\
&\mathbf{U}^0{}_1 + {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_1 + 7\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_1}^5 - 14\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_2 + \mathbf{U}^0{}_2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^3 + 12\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^2 \\
&\mathbf{U}^0{}_2 + 2\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1 + \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^2 + 10\, {\mathbf{U}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^5 + 8\, {\mathbf{Y}^0{}_1}^4\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_2}^2 - 6\, {\mathbf{U}^0{}_3}^2\, {\mathbf{Y}^0{}_2}^3\, {\mathbf{Y}^0{}_1}^2)\, \Omega^2 + 2 \\
&{\mathbf{Y}^0{}_1}^4\, \mathbf{Y}^0{}_2\, (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\, (-\mathbf{U}^0{}_1\, \mathbf{Y}^0{}_2 + \mathbf{U}^0{}_2\, \mathbf{Y}^0{}_1)\, (\mathbf{Y}^0{}_1\, \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2)\bigg)\, \mathbf{Y}^1{}_2 + \bigg(-6\, {\mathbf{Y}^0{}_2}^5\, {\mathbf{U}^0{}_1}^3 + {\mathbf{Y}^0{}_1}^4\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2 - 2 \\
&\mathbf{U}^0{}_3\, {\mathbf{U}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^5 + 5\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{U}^0{}_2}^3 - 5\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_1}^5 - \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_1}^6\, \mathbf{U}^0{}_3 + 8\, {\mathbf{Y}^0{}_2}^5\, {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3 - {\mathbf{Y}^0{}_1}^4\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{U}^0{}_2}^2 + {\mathbf{Y}^0{}_1}^3 \\
&{\mathbf{Y}^0{}_2}^3\, {\mathbf{U}^0{}_1}^2 - {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^3\, {\mathbf{U}^0{}_2}^2 - 4\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_2}^3\, {\mathbf{Y}^0{}_1}^4 + 6\, {\mathbf{Y}^0{}_2}^4\, {\mathbf{U}^0{}_1}^3\, \mathbf{Y}^0{}_1 - \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^4\, {\mathbf{U}^0{}_1}^3 - 6\, \mathbf{Y}^0{}_1\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^4\, {\mathbf{U}^0{}_2}^2 + 12 \\
&{\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{U}^0{}_1}^2 - 5\, {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_3\, \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_1}^2 - {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^3\, {\mathbf{U}^0{}_2}^2 - {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 - 19\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^3 \\
&\mathbf{U}^0{}_2 + {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_2 + 4\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^5 + 14\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{U}^0{}_2}^2 - {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2\, {\mathbf{U}^0{}_1}^2 + {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^4 \\
&\mathbf{U}^0{}_2 - {\mathbf{Y}^0{}_1}^5\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1 - 6\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_2}^2 + 9\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3 + 12\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_3\, \mathbf{Y}^0{}_1 - 18\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{U}^0{}_2}^2 \\
&\mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^3 + 18\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_2\, {\mathbf{U}^0{}_1}^2 + {\mathbf{Y}^0{}_1}^5\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_3 - 5\, {\mathbf{Y}^0{}_1}^5\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_3 - 6\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_3 - 2\, {\mathbf{Y}^0{}_2}^5\, \mathbf{Y}^0{}_1 \\
&\mathbf{U}^0{}_1\, \mathbf{U}^0{}_3 - 2\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_3 + 5\, {\mathbf{Y}^0{}_1}^4\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_3 + 2\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_3 - 2\, {\mathbf{Y}^0{}_1}^4\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_3 + 2\, {\mathbf{Y}^0{}_1}^3 \\
&\mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3 - 3\, \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_2\, {\mathbf{U}^0{}_1}^2 - \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_1}^4\, {\mathbf{U}^0{}_2}^2 - 17\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^4\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2 - 4\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_2}^4\, \mathbf{U}^0{}_3 \\
&\mathbf{Y}^0{}_1 - 17\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^3 + 4\, \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_1}^6\, \mathbf{U}^0{}_3\bigg)\, \Omega^3 + \bigg(6\, {\mathbf{U}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^4 + 6\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{U}^0{}_2}^2 + {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_2\, \mathbf{U}^0{}_1 - {\mathbf{Y}^0{}_1}^3 \\
&\mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_1}^2 - 6\, {\mathbf{U}^0{}_1}^2\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3 + {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2 - 12\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_2 + 6\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_1\, \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^3 - 4\, \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_1}^3\, \mathbf{Y}^0{}_2 \\
&\mathbf{U}^0{}_3 + {\mathbf{Y}^0{}_1}^2\, {\mathbf{U}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2 + 5\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_1}^2 + 11\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 + 2\, \mathbf{U}^0{}_1\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3 - 3\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^2 \\
&\mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_2}^2 - \mathbf{Y}^0{}_2\, {\mathbf{U}^0{}_2}^3\, {\mathbf{Y}^0{}_1}^2 + {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_1\, {\mathbf{U}^0{}_2}^2 - \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{U}^0{}_2}^2 - 2\, \mathbf{Y}^0{}_1\, {\mathbf{U}^0{}_3}^3\, {\mathbf{Y}^0{}_2}^2 - 4\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_1}^4 - 5\, {\mathbf{U}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^3 \\
&\mathbf{Y}^0{}_2 + 2\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{U}^0{}_3}^3\, \mathbf{Y}^0{}_2 - 2\, {\mathbf{Y}^0{}_1}^3\, {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3 + 2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_1 + 6\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3\, {\mathbf{U}^0{}_2}^2 - 4\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{U}^0{}_1}^2\, \mathbf{U}^0{}_3 \\
&\mathbf{Y}^0{}_2 + {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_1\, \mathbf{U}^0{}_3\, \mathbf{U}^0{}_2\bigg)\, \Omega^5 + \bigg(4\, \Omega^7\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3\, \mathbf{Y}^0{}_1 + \mathbf{Y}^0{}_2\, (6\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 - 5\, {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2 + {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_1 - 6\, {\mathbf{Y}^0{}_1}^2 \\
&\mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3 + 6\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_1 - 6\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1 - 5\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3)\, \Omega^5 + \mathbf{Y}^0{}_1\, \mathbf{Y}^0{}_2\, (3\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3\, \mathbf{Y}^0{}_2 - {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 + 9\, {\mathbf{Y}^0{}_2}^2 \\
&{\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3 - \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_2}^3\, \mathbf{Y}^0{}_1 + 5\, {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_3 - {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^2 - \mathbf{U}^0{}_1\, \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^3 + 3\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^4 + 2\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3)\, \Omega^3\bigg)\, \mathbf{E}_2 \\
&+\bigg(\Omega^8\, \mathbf{Y}^0{}_1\, \mathbf{U}^0{}_3 + (2\, {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3 + 4\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1 - 4\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 + 4\, {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2 + {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3 - 4\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_1) \\
&\Omega^6 - \mathbf{Y}^0{}_1\, (\mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^3 + 2\, {\mathbf{Y}^0{}_2}^4 + 5\, {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2 + 2\, {\mathbf{Y}^0{}_1}^4)\, \mathbf{U}^0{}_3\, \Omega^4\bigg)\, \mathbf{U}^1{}_3 + \bigg(-4\, \Omega^7\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1\, (6\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 - 5 \\
&{\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2 + {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_1 - 6\, {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3 + 6\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_1 - 6\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_1 - 5\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3)\, \Omega^5 + {\mathbf{Y}^0{}_1}^2\, (3\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3 \\
&\mathbf{Y}^0{}_2 - {\mathbf{Y}^0{}_1}^2\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 + 9\, {\mathbf{Y}^0{}_2}^2\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3 - \mathbf{U}^0{}_2\, {\mathbf{Y}^0{}_2}^3\, \mathbf{Y}^0{}_1 + 5\, {\mathbf{Y}^0{}_1}^4\, \mathbf{U}^0{}_3 - {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_1\, {\mathbf{Y}^0{}_2}^2 - \mathbf{U}^0{}_1\, \mathbf{Y}^0{}_2\, {\mathbf{Y}^0{}_1}^3 + 3\, \mathbf{U}^0{}_3 \\
&{\mathbf{Y}^0{}_2}^4 + 2\, \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3)\, \Omega^3\bigg)\, \mathbf{E}_1 + \bigg((\mathbf{Y}^0{}_1\, (-2\, \mathbf{U}^0{}_3 + \mathbf{U}^0{}_2)\, \Omega^7 + \mathbf{Y}^0{}_1\, (-\mathbf{Y}^0{}_2\, \mathbf{U}^0{}_2\, \mathbf{Y}^0{}_1 + 8\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_3 + 4\, {\mathbf{Y}^0{}_1}^2\, \mathbf{U}^0{}_3 - \mathbf{U}^0{}_1 \\
&\mathbf{Y}^0{}_2\, \mathbf{Y}^0{}_1 - 2\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 + 9\, \mathbf{U}^0{}_3\, \mathbf{Y}^0{}_1\, \mathbf{Y}^0{}_2)\, \Omega^5 - \mathbf{Y}^0{}_1\, \mathbf{Y}^0{}_2\, (7\, {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_2 - {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_2 - {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2 \\
&\mathbf{U}^0{}_1 + 6\, {\mathbf{Y}^0{}_2}^3\, \mathbf{U}^0{}_3 + 8\, {\mathbf{Y}^0{}_1}^3\, \mathbf{U}^0{}_3 + 9\, \mathbf{Y}^0{}_1\, \mathbf{U}^0{}_3\, {\mathbf{Y}^0{}_2}^2 - \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, \mathbf{U}^0{}_1)\, \Omega^3\bigg)\, \mathbf{Y}^1{}_2 + ({\mathbf{Y}^0{}_1}^2\, \Omega^7 - {\mathbf{Y}^0{}_1}^2\, \mathbf{Y}^0{}_2\, (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1) \\
&\Omega^5)\, \mathbf{U}^1{}_1 + (\mathbf{Y}^0{}_1\, \mathbf{Y}^0{}_2\, \Omega^7 - \mathbf{Y}^0{}_1\, {\mathbf{Y}^0{}_2}^2\, (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\, \Omega^5)\, \mathbf{U}^1{}_2 + (-\mathbf{Y}^0{}_1\, (\mathbf{Y}^0{}_2 + 2\, \mathbf{Y}^0{}_1)\, \Omega^7 + {\mathbf{Y}^0{}_1}^2\, (\mathbf{Y}^0{}_1\, \mathbf{Y}^0{}_2 + 2\, {\mathbf{Y}^0{}_1}^2 + 3
\end{aligned}
$$

$$\mathbf{Y}^0{}_2{}^2)\ \Omega^5)\ \mathbf{U}^1{}_3 + (\mathbf{Y}^0{}_1\ (-8\ \mathbf{Y}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2)\ \Omega^6 - \mathbf{Y}^0{}_1{}^3\ (\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_2{}^2)\ \Omega^4)\ \mathbf{E}_1 + \Big(\mathbf{Y}^0{}_2\ (3\ \mathbf{Y}^0{}_1$$
$$\mathbf{Y}^0{}_2 - 4\ \mathbf{Y}^0{}_2{}^2 + 6\ \mathbf{Y}^0{}_1{}^2)\ \Omega^6 - \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^2\ (\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_2{}^2)\ \Omega^4\Big)\ \mathbf{E}_2 + (\Omega^8\ \mathbf{Y}^0{}_1 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ \Omega^6)$$
$$\mathbf{E}_3 + 4\ \Omega^8\ \mathbf{Y}^0{}_2{}^2\ \frac{\partial \mathbf{E_1}}{\partial x_1} - 4\ \Omega^8\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ \frac{\partial \mathbf{E_2}}{\partial x_1} - \mathbf{Y}^0{}_1\ (-\mathbf{U}^0{}_2 + 2\ \mathbf{U}^0{}_1)\ \Omega^8 + \Big(5\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 - 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2$$
$$\mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^3 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1 - 2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3{}^2 - 6\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1 - 8\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2 - 9$$
$$\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + 4\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2\Big)\ \Omega^6 + \Big(-6\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^2 + \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3{}^2 - 6\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1 + 14\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1$$
$$\mathbf{U}^0{}_2 + 4\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 + 2\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^4 - 13\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^4 - 2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + 3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - 13\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3{}^2$$
$$\mathbf{Y}^0{}_1{}^3 - 8\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 + \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 - 6\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 + 4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2 + \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3\Big)\ \Omega^4 + \Big(3$$
$$\mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2 + 4\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1{}^2 - 5\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 + 7\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1$$
$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 + 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2 + 6\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2$$
$$\mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 + 2\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 18\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^3 + 6\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^5 - 6\ \mathbf{Y}^0{}_1{}^6\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 - 5\ \mathbf{Y}^0{}_1$$
$$\mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^5 + 5\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2{}^2 + 7\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4 - 11\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 - 8\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^6 + 7\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3{}^2 + 21$$
$$\mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^2 + 13\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3{}^2 + 16\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2{}^2 - 18\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_2{}^6\ \mathbf{U}^0{}_2{}^2 - 6\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^6 + 8$$
$$\mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^6 + 13\ \mathbf{Y}^0{}_1{}^6\ \mathbf{U}^0{}_3{}^2)\ \Omega^2 + 2\ \mathbf{Y}^0{}_1{}^5\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ (-\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2 + \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\Big)\ \mathbf{Y}^1{}_1 - 4\ \Omega^9\ \mathbf{Y}^0{}_1$$
$$\frac{\partial \mathbf{E_3}}{\partial t} - 2\ \mathbf{U}^0{}_3\ (\mathbf{Y}^0{}_1 + 2\ \mathbf{U}^0{}_2)\ \Omega^9 + \mathbf{U}^0{}_3\ \Big(-5\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_2{}^2 - 3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1 + 4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\Big)\ \Omega^7 - 2\ \mathbf{Y}^0{}_1\ \Big(-3$$
$$\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1 - \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_2\Big)\ \Omega^7\ \frac{\partial \mathbf{E_3}}{\partial x_2} + 6\ \Omega^7\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2\ \frac{\partial \mathbf{E_2}}{\partial x_1} + 6\ \Omega^7\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^2\ \frac{\partial \mathbf{E_2}}{\partial x_2} + 2\ \mathbf{Y}^0{}_1\ (-2$$
$$\mathbf{Y}^0{}_2{}^2 + \mathbf{Y}^0{}_1{}^2)\ \mathbf{U}^0{}_3\ \Omega^7\ \frac{\partial \mathbf{E_1}}{\partial x_1} + 2\ \mathbf{Y}^0{}_1\ \Big(-2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 + 3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1\Big)\ \Omega^7\ \frac{\partial \mathbf{E_3}}{\partial x_1} + 2\ \Omega^9\ \mathbf{Y}^0{}_1\ \frac{\partial \mathbf{E_3}}{\partial x_3}\ \mathbf{U}^0{}_3 - 2$$
$$(-2\ \mathbf{Y}^0{}_2{}^2 + \mathbf{Y}^0{}_1{}^2)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^7\ \frac{\partial \mathbf{E_1}}{\partial x_3} - 6\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^7\ \frac{\partial \mathbf{E_2}}{\partial x_3} + 2\ \mathbf{Y}^0{}_2\ (-2\ \mathbf{Y}^0{}_2{}^2 + \mathbf{Y}^0{}_1{}^2)$$
$$\mathbf{U}^0{}_3\ \Omega^7\ \frac{\partial \mathbf{E_1}}{\partial x_2} + (\mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \Omega^5\ \mathbf{Y}^0{}_1 - \Omega^7\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3)\ \mathbf{Y}^1{}_2{}^2\Big)\ \Big/\ \Omega^{10},$$

$$\frac{d\mathbf{U}^1{}_2}{dt} = 1/4\ \Big(-4\ \Omega^9\ \mathbf{Y}^0{}_2\ \frac{\partial \mathbf{E_3}}{\partial t} - 2\ \mathbf{U}^0{}_3\ (\mathbf{Y}^0{}_2 - 4\ \mathbf{U}^0{}_1)\ \Omega^9 - \mathbf{U}^0{}_3\ \Big(3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2 + 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 - 4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_2$$
$$\mathbf{U}^0{}_2{}^2 - 2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_2 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1 + 4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2\Big)\ \Omega^7 + 2\ (-\mathbf{Y}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1{}^2)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^7\ \frac{\partial \mathbf{E_2}}{\partial x_3} + 2\ \Omega^9$$
$$\mathbf{Y}^0{}_2\ \frac{\partial \mathbf{E_3}}{\partial x_3}\ \mathbf{U}^0{}_3 + 2\ \mathbf{Y}^0{}_2\ \big(-2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 + 3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1\big)\ \Omega^7\ \frac{\partial \mathbf{E_3}}{\partial x_1} - 2\ \mathbf{Y}^0{}_2\ \big(-3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1 - \mathbf{Y}^0{}_2{}^2$$
$$\mathbf{U}^0{}_2 + 2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_2\big)\ \Omega^7\ \frac{\partial \mathbf{E_3}}{\partial x_2} + \Big(-2\ \mathbf{Y}^0{}_1\ (-2\ \mathbf{Y}^0{}_2{}^2 + \mathbf{Y}^0{}_1{}^2)\ \Omega^6\ \mathbf{E}_1 - 6\ \Omega^6\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{E}_2 - 4\ \Omega^8\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2\ \frac{\partial \mathbf{E_1}}{\partial x_2} + 4\ \Omega^8$$
$$\mathbf{Y}^0{}_1{}^2\ \frac{\partial \mathbf{E_2}}{\partial x_2} + (-2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2)\ \Omega^8 + (-2\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2 - 6\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2 + 4\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^3 - 2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3{}^2 - \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1 - 2$$
$$\mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + 3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3{}^2 + \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 - 11\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + 3\ \mathbf{U}^0{}_1{}^2$$
$$\mathbf{Y}^0{}_2{}^2)\ \Omega^6 + (-4\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^4 - \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - 8\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2 + 10\ \mathbf{U}^0{}_1{}^2$$
$$\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3 - 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 - 2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3{}^2 + \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 + 5\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 - 7\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3{}^2$$
$$\mathbf{Y}^0{}_1 + \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1{}^2 - \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^3 + 2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 + 7\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1{}^2 - 11\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3$$
$$\mathbf{Y}^0{}_2 + \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^4 + \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^2 + 2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2)\ \Omega^4 + \mathbf{Y}^0{}_2\ (7\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^4 - 18\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_1 - 14$$
$$\mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2 - 6\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 + 18\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^5 - \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1 - 20\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1{}^2$$
$$\mathbf{Y}^0{}_2{}^2 - 5\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^2 + \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^5 - 16\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 31\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + 6\ \mathbf{Y}^0{}_2{}^4$$
$$\mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1 - 2\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 + 9\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1{}^2 + 7\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2{}^2 - 8\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^5 + 7\ \mathbf{Y}^0{}_1{}^3$$
$$\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1 + 7\ \mathbf{U}^0{}_3{}^2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2 + 26\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^2)\ \Omega^2 + 2$$
$$\mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ (-\mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2 + \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1)\Big)\ \mathbf{Y}^1{}_2 + \Big(\mathbf{Y}^0{}_2\ \Omega^8\ \mathbf{U}^0{}_3 - (-\mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2 - 2\ \mathbf{Y}^0{}_2$$
$$\mathbf{Y}^0{}_1{}^2 + \mathbf{Y}^0{}_1{}^3 - \mathbf{Y}^0{}_2{}^3)\ \mathbf{U}^0{}_3\ \Omega^6 - \mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + 2\ \mathbf{Y}^0{}_2{}^4 + 5\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1{}^4)\ \mathbf{U}^0{}_3\ \Omega^4\Big)\ \mathbf{U}^1{}_3 + \Big(\mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^5 + 2$$
$$\mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2{}^3 - \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1{}^2 - \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 - 5\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2{}^2 - 8\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^2$$
$$\mathbf{U}^0{}_1{}^2 - 15\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 5\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^2 - \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1$$
$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2 - \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1{}^2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2 - 3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 - 11\ \mathbf{U}^0{}_1{}^2$$
$$\mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 - 2\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1 - 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_2{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^3 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 4\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^5$$
$$\mathbf{Y}^0{}_2\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - 2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - 2\ \mathbf{Y}^0{}_2{}^5\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 - 6\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 + 2\ \mathbf{Y}^0{}_1{}^2$$
$$\mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 - 5\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3 + 5\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 - 6\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 + 10\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3$$
$$\mathbf{Y}^0{}_1 + 9\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_2{}^5 - 2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^6\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^2$$
$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_2{}^3 - \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1{}^3 + \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_3 - 31\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3$$
$$\mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^2\Big)\ \Omega^3 + \big(-\mathbf{Y}^0{}_2\ \Omega^7\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\ \Omega^5\big)\ \mathbf{Y}^1{}_2{}^2 + \Big(4\ \Omega^8\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_2\ (4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_2 - 4\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + 4\ \mathbf{U}^0{}_1$$
$$\mathbf{Y}^0{}_2{}^2 - 4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1 + \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2)\ \Omega^6 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 + \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3)\ \Omega^4 - \mathbf{Y}^0{}_1{}^4$$
$$\mathbf{Y}^0{}_2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^2\Big)\ \mathbf{U}^1{}_2 + \Big(-4\ \mathbf{Y}^0{}_2\ \Omega^8\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_1{}^2\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^6 + \mathbf{Y}^0{}_2$$
$$\mathbf{Y}^0{}_1{}^2\ (\mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 + \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1 + \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3)\ \Omega^4 + \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^2\ (\mathbf{Y}^0{}_2 + \mathbf{Y}^0{}_1)\ (\mathbf{Y}^0{}_1\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2)\ \Omega^2\Big)$$
$$\mathbf{U}^1{}_1 + \Big(\mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_2{}^3 - 3\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^7 - 2\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^7 + \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^5\ \mathbf{U}^0{}_1{}^3 + 10\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3 + 17$$
$$\mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 + 3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_2{}^3 + 17\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^6\ \mathbf{Y}^0{}_2 + 17\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^6\ \mathbf{Y}^0{}_1 + 21\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^4$$
$$\mathbf{Y}^0{}_2{}^3 + 33\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{Y}^0{}_1{}^3 + 3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2{}^2\ \mathbf{U}^0{}_1 + 15\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1{}^2 + 3\ \mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1{}^2 + 6\ \mathbf{Y}^0{}_2{}^6$$
$$\mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1 + 3\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2 + 12\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^5\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + 11\ \mathbf{Y}^0{}_1{}^5\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 - 3\ \mathbf{Y}^0{}_1{}^6\ \mathbf{Y}^0{}_2$$
$$\mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1 + 11\ \mathbf{Y}^0{}_1{}^4\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 + 11\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_2{}^7\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^6\ \mathbf{U}^0{}_2{}^3 + \mathbf{Y}^0{}_2{}^3$$
$$\mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1{}^3 + 3\ \mathbf{Y}^0{}_2{}^5\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2{}^2 + 3\ \mathbf{Y}^0{}_2{}^4\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_2 - 2\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^6\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{U}^0{}_1\Big)\ \Omega + \Big(2\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_2{}^2 - 3$$
$$\mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1 - \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1{}^2 + 11\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^2 - 4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2$$
$$\mathbf{U}^0{}_1{}^2\ \mathbf{U}^0{}_2 - 3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1 + \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 - 13\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 - 4\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_1{}^2 + \mathbf{U}^0{}_3$$
$$\mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1{}^4 - 7\ \mathbf{U}^0{}_3\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2{}^2 + 2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3{}^3\ \mathbf{Y}^0{}_2 - 2\ \mathbf{Y}^0{}_1\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 - 2\ \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_3{}^3 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^3 + 3$$
$$\mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_1{}^3\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1\ \mathbf{U}^0{}_2 - \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_3\ \mathbf{U}^0{}_2{}^2 + \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2{}^2 - 2\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1$$
$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3\Big)\ \Omega^5 + \Big(4\ \mathbf{Y}^0{}_1{}^2\ \Omega^8\ \frac{\partial \mathbf{E_2}}{\partial x_3} - 4\ \mathbf{Y}^0{}_1\ \Omega^8\ \frac{\partial \mathbf{E_1}}{\partial x_3}\ \mathbf{Y}^0{}_2\Big)\ \mathbf{Y}^1{}_3 + \Big(4\ \Omega^9\ \mathbf{Y}^0{}_1 - \mathbf{Y}^0{}_1\ (\mathbf{Y}^0{}_2\ \mathbf{U}^0{}_3 - 4\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2 + 4\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_1)$$
$$\Omega^7 + (-3\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3 + 5\ \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^3\ \mathbf{Y}^0{}_1 - \mathbf{U}^0{}_2\ \mathbf{Y}^0{}_2{}^4 + 3\ \mathbf{Y}^0{}_1\ \mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_1 - 6\ \mathbf{Y}^0{}_2{}^4\ \mathbf{U}^0{}_1 - 4\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2\ \mathbf{U}^0{}_2 + \mathbf{Y}^0{}_1$$
$$\mathbf{Y}^0{}_2{}^3\ \mathbf{U}^0{}_3 - \mathbf{Y}^0{}_1{}^4\ \mathbf{U}^0{}_1 + \mathbf{Y}^0{}_2{}^2\ \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_3 + \mathbf{Y}^0{}_1{}^2\ \mathbf{U}^0{}_1\ \mathbf{Y}^0{}_2{}^2)\ \Omega^5 + \mathbf{Y}^0{}_2\ (3\ \mathbf{Y}^0{}_1{}^2\ \mathbf{Y}^0{}_2{}^2 + \mathbf{Y}^0{}_2{}^4 + \mathbf{Y}^0{}_1{}^4 + \mathbf{Y}^0{}_2\ \mathbf{Y}^0{}_1{}^3)\ (\mathbf{Y}^0{}_1$$

$${\mathbf{U}^0}_1+{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2)\ \Omega^3\Big)\ \mathbf{E}_3\ +\Big(4\ \Omega^7\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_1+{\mathbf{Y}^0}_1\ (-8\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2-3\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^2+3\ {\mathbf{Y}^0}_1^3$$
$${\mathbf{U}^0}_3-3\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3+{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1)\ \Omega^5+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ (3\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2-{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+9\ {\mathbf{Y}^0}_2^2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3-{\mathbf{U}^0}_2$$
$${\mathbf{Y}^0}_2^3\ {\mathbf{Y}^0}_1+5\ {\mathbf{Y}^0}_1^4\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^3+3\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^4+2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3)\ \Omega^3\Big)\ \mathbf{E}_1\ +\Big(-{\mathbf{Y}^0}_2\ \Omega^7$$
$${\mathbf{U}^0}_3+{\mathbf{Y}^0}_2\ (8\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1+5\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2+2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3)\ \Omega^5-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ (8\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3$$
$${\mathbf{Y}^0}_2^2-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_1+7\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+6\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3+7\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1-{\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2)$$
$$\Omega^3\Big)\ {\mathbf{Y}^1}_1^2+\Big((\,{\mathbf{Y}^0}_2\ (-2\ {\mathbf{U}^0}_3+{\mathbf{U}^0}_2)\ \Omega^7+{\mathbf{Y}^0}_2\ (-{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1+8\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3+4\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1-2\ {\mathbf{Y}^0}_2^2$$
$${\mathbf{U}^0}_2+9\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2)\ \Omega^5-{\mathbf{Y}^0}_2^2\ (7\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2-{\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2-{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1+6\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3+8$$
$${\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3+9\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^2-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_1)\ \Omega^3)\ {\mathbf{Y}^1}_2+({\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ \Omega^7-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ \Omega^5)\ {\mathbf{U}^1}_1+({\mathbf{Y}^0}_2^2$$
$$\Omega^7-{\mathbf{Y}^0}_2^3\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ \Omega^5)\ {\mathbf{U}^1}_2+(-{\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_2+2\ {\mathbf{Y}^0}_1)\ \Omega^7+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2+2\ {\mathbf{Y}^0}_1^2+3\ {\mathbf{Y}^0}_2^2)\ \Omega^5)\ {\mathbf{U}^1}_3+\big((6$$
$${\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^2-2\ {\mathbf{Y}^0}_1^3-4\ {\mathbf{Y}^0}_2^3+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2)\ \Omega^6-{\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^2\ ({\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2+2\ {\mathbf{Y}^0}_1^2+3\ {\mathbf{Y}^0}_2^2)\ \Omega^4\big)\ \mathbf{E}_1\ +\big(-{\mathbf{Y}^0}_2\ (-10\ {\mathbf{Y}^0}_1$$
$${\mathbf{Y}^0}_2+2\ {\mathbf{Y}^0}_1^2-{\mathbf{Y}^0}_2^2)\ \Omega^6-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ ({\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2+2\ {\mathbf{Y}^0}_1^2+3\ {\mathbf{Y}^0}_2^2)\ \Omega^4\big)\ \mathbf{E}_2\ +(\Omega^8\ {\mathbf{Y}^0}_2-{\mathbf{Y}^0}_2^2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ \Omega^6)\ \mathbf{E}_3\ -4\ \Omega^8$$
$${\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ \frac{\partial \mathbf{E_1}}{\partial x_1}\ +4\ \Omega^8\ {\mathbf{Y}^0}_1^2\ \frac{\partial \mathbf{E_2}}{\partial x_1}\ +({\mathbf{U}^0}_1^2+{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2-2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2)\ \Omega^8+\Big({\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1-2\ {\mathbf{U}^0}_2^2\ {\mathbf{Y}^0}_1^2-2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1^2-{\mathbf{U}^0}_1$$
$${\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1+{\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_2^2-{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2-2\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2-2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1^3+3\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2^2+15\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1^2+3$$
$${\mathbf{U}^0}_2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2+4\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1+2\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\Big)\ \Omega^6+\Big({\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2^2-10\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1-7\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1$$
$${\mathbf{U}^0}_2+6\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2^2-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_2+11\ {\mathbf{U}^0}_2^2\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2-8\ {\mathbf{Y}^0}_1^4\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_1-6\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_3^2-19\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2$$
$${\mathbf{U}^0}_2+2\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1^4+{\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_1^2+2\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2+{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2-11\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1^2-16$$
$${\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1^3)\ \Omega^4+\big(-3\ {\mathbf{Y}^0}_1^4\ {\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_2^2+3\ {\mathbf{Y}^0}_1^4\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2^2-10\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1^2-16\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2^2+6\ {\mathbf{Y}^0}_1^4$$
$${\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+5\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2+11\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_2+8\ {\mathbf{Y}^0}_1^5\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_1-{\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_1^2$$
$${\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_1^4\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_1\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3-12\ {\mathbf{Y}^0}_1^5\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1^2+21\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_2^3+7\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3^2$$
$${\mathbf{Y}^0}_2^5+8\ {\mathbf{Y}^0}_1^6\ {\mathbf{U}^0}_1\ {\mathbf{U}^0}_2-2\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_2^5-5\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^5\ {\mathbf{U}^0}_2^2-3\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_2^4-6\ {\mathbf{Y}^0}_1^5\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2^2+4\ {\mathbf{U}^0}_1\ {\mathbf{U}^0}_2$$
$${\mathbf{Y}^0}_2^6+18\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_3^2+13\ {\mathbf{Y}^0}_1^4\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_2^2+13\ {\mathbf{Y}^0}_1^5\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3^2+4\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_2^2-4\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^5\ {\mathbf{U}^0}_2+3$$
$${\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_2^6+6\ {\mathbf{Y}^0}_2^6\ {\mathbf{U}^0}_3^2+3\ {\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_1^6)\ \Omega^2+2\ {\mathbf{Y}^0}_1^4\ {\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ (-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2+{\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1)\ ({\mathbf{Y}^0}_1\ {\mathbf{U}^0}_1+{\mathbf{Y}^0}_2$$
$${\mathbf{U}^0}_2)\Big)\ {\mathbf{Y}^1}_1+\Big(-4\ \Omega^7\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3+{\mathbf{Y}^0}_2\ (-8\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2-3\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^2+3\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3-3\ {\mathbf{Y}^0}_2^3$$
$${\mathbf{U}^0}_3+{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1)\ \Omega^5+{\mathbf{Y}^0}_2^2\ (3\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2-{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+9\ {\mathbf{Y}^0}_2^2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3-{\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2^3\ {\mathbf{Y}^0}_1+5\ {\mathbf{Y}^0}_1^4$$
$${\mathbf{U}^0}_3-{\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^3+3\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^4+2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3)\ \Omega^3\Big)\ \mathbf{E}_2\ +6\ \Omega^7\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ \frac{\partial \mathbf{E_1}}{\partial x_2}\ {\mathbf{U}^0}_3+6\ \Omega^7\ {\mathbf{Y}^0}_1^2$$
$${\mathbf{Y}^0}_2\ \frac{\partial \mathbf{E_1}}{\partial x_1}\ {\mathbf{U}^0}_3-6\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_1\ {\mathbf{U}^0}_1+{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2)\ \Omega^7\ \frac{\partial \mathbf{E_1}}{\partial x_3}\ -2\ {\mathbf{Y}^0}_1\ (-{\mathbf{Y}^0}_2^2+2\ {\mathbf{Y}^0}_1^2)\ {\mathbf{U}^0}_3\ \Omega^7\ \frac{\partial \mathbf{E_2}}{\partial x_1}\ -2\ {\mathbf{Y}^0}_2\ (-{\mathbf{Y}^0}_2^2+2$$
$${\mathbf{Y}^0}_1^2)\ {\mathbf{U}^0}_3\ \Omega^7\ \frac{\partial \mathbf{E_2}}{\partial x_2}\ \Big)\ \Big/\ \Omega^{10},$$

$$\frac{d{\mathbf{U}^1}_3}{dt}\ =\ 1/4\ \Big(4\ \Omega^7\ {\mathbf{Y}^0}_2\ \frac{\partial \mathbf{E_2}}{\partial t}\ +4\ \Omega^7\ \frac{\partial \mathbf{E_1}}{\partial t}\ {\mathbf{Y}^0}_1+2\ {\mathbf{Y}^0}_1\ \Big(-3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1-{\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_2\Big)\ \Omega^5\ \frac{\partial \mathbf{E_1}}{\partial x_2}\ +2\ \Omega^7$$
$${\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2\ \frac{\partial \mathbf{E_3}}{\partial x_2}\ -2\ \Omega^7\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_1\ \frac{\partial \mathbf{E_1}}{\partial x_3}\ +2\ \Omega^7\ {\mathbf{U}^0}_3\ \frac{\partial \mathbf{E_3}}{\partial x_1}\ {\mathbf{Y}^0}_1-2\ \Omega^7\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2\ \frac{\partial \mathbf{E_2}}{\partial x_3}\ -2\ {\mathbf{Y}^0}_2\ \Big(-2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2+{\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1+3$$
$${\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1\Big)\ \Omega^5\ \frac{\partial \mathbf{E_2}}{\partial x_1}\ +2\ {\mathbf{Y}^0}_2\ \Big(-3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1-{\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_2\Big)\ \Omega^5\ \frac{\partial \mathbf{E_2}}{\partial x_2}\ -2\ {\mathbf{Y}^0}_1\ \Big(-2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2+{\mathbf{Y}^0}_1^2$$
$${\mathbf{U}^0}_1+3\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1\Big)\ \Omega^5\ \frac{\partial \mathbf{E_1}}{\partial x_1}\ +\Big(-2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1^5\ {\mathbf{U}^0}_3^2-2\ {\mathbf{Y}^0}_2^5\ {\mathbf{U}^0}_1^3-2\ {\mathbf{Y}^0}_2^2\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_2^3+2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2^3\ {\mathbf{Y}^0}_1^4+8\ {\mathbf{Y}^0}_2^4$$
$${\mathbf{U}^0}_1^3\ {\mathbf{Y}^0}_1+{\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_2^2+{\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_1^2+{\mathbf{Y}^0}_1^4\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1^2+{\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2^2-8\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1^2$$
$${\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2+2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2^2+4\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_1^2-{\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2^2-{\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3-{\mathbf{U}^0}_1^2$$
$${\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_1+2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_2^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^3+2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_1^2-2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_1^4\ {\mathbf{U}^0}_2^2+4\ {\mathbf{U}^0}_1\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^2$$
$${\mathbf{Y}^0}_1^3+4\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2^3+4\ {\mathbf{Y}^0}_2^2\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_1^3-{\mathbf{Y}^0}_1^4\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2^2-3\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^2-5\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3^2$$
$${\mathbf{Y}^0}_1^4+2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3^2+2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^4\ {\mathbf{U}^0}_3^2-3\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1-3\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_2^3\ {\mathbf{Y}^0}_1^2-6\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_1$$
$${\mathbf{Y}^0}_1\ {\mathbf{U}^0}_2^2+4\ {\mathbf{U}^0}_1\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_1^2-2\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1^3\ {\mathbf{Y}^0}_1^2+2\ {\mathbf{Y}^0}_2^5\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_1^2\Big)\ \Omega+\Big(4\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_1-4\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2$$
$${\mathbf{U}^0}_1^2+2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3^2+2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2-2\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1\ {\mathbf{U}^0}_1^2\ {\mathbf{U}^0}_3+{\mathbf{U}^0}_3\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_2^2+4\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2^2-4$$
$${\mathbf{U}^0}_2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2\Big)\ \Omega^5+\Big({\mathbf{U}^0}_3\ (-3\ {\mathbf{U}^0}_2+{\mathbf{Y}^0}_1)\ \Omega^6-{\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3\ (2\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_1+3\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2)\ \Omega^4+{\mathbf{U}^0}_3\ (-{\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^3+7$$
$${\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2+4\ {\mathbf{Y}^0}_1^4\ {\mathbf{U}^0}_1-3\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1+3\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1^4+10\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2^3\ {\mathbf{Y}^0}_1+4\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2^4)$$
$$\Omega^2+{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3\ \big(-{\mathbf{Y}^0}_2^5\ {\mathbf{U}^0}_2-{\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+{\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^3+2\ {\mathbf{Y}^0}_1^5\ {\mathbf{U}^0}_2+2\ {\mathbf{Y}^0}_1^4\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_2-3$$
$${\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_1-3\ {\mathbf{Y}^0}_1^4\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1-5\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_1\big)\Big)\ {\mathbf{Y}^1}_2+\Big(4\ \Omega^7\ {\mathbf{Y}^0}_1+2\ {\mathbf{Y}^0}_1\ (6\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1+{\mathbf{Y}^0}_2$$
$${\mathbf{U}^0}_3-6\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2)\ \Omega^5-2\ {\mathbf{Y}^0}_2^2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ ({\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3+{\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2)\ \Omega^3+2\ {\mathbf{Y}^0}_2^2\ ({\mathbf{Y}^0}_1^3+2\ {\mathbf{Y}^0}_2^3+{\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^2)$$
$$(-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2+{\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1)\ \Omega\Big)\ \mathbf{E}_2\ +\Big(((-2\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_1+3\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2+3\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1)\ \Omega^5+(-2\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2-6\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2-5$$
$${\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2-2\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1+2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_1-3\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1^3)\ \Omega^3+{\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^2+{\mathbf{Y}^0}_1^3-{\mathbf{Y}^0}_2^3+3\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2)$$
$$({\mathbf{Y}^0}_1\ {\mathbf{U}^0}_1+{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2)\ \Omega\big)\ {\mathbf{Y}^1}_2+({\mathbf{Y}^0}_2\ \Omega^5\ {\mathbf{Y}^0}_1-{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ \Omega^3)\ {\mathbf{U}^1}_1+(\Omega^5\ {\mathbf{Y}^0}_2^2-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)$$
$$\Omega^3)\ {\mathbf{U}^1}_2+(\Omega^7-{\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ \Omega^5)\ {\mathbf{U}^1}_3+(-{\mathbf{Y}^0}_1\ \Omega^6+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ \Omega^4)\ \mathbf{E}_1\ +(-{\mathbf{Y}^0}_2\ \Omega^6+{\mathbf{Y}^0}_2^2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)$$
$$\Omega^4)\ \mathbf{E}_2\ +({\mathbf{Y}^0}_2\ \Omega^6-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2\ ({\mathbf{Y}^0}_2+{\mathbf{Y}^0}_1)\ \Omega^4)\ \mathbf{E}_3\ +\big({\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2-4\ {\mathbf{U}^0}_1\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3-{\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1-{\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3\big)\ \Omega^6+\big(3$$
$${\mathbf{U}^0}_3\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_2^2-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_1+{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2+2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1+{\mathbf{Y}^0}_1^3\ {\mathbf{U}^0}_3-{\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_2^2-{\mathbf{U}^0}_2^2\ {\mathbf{Y}^0}_1^2+{\mathbf{Y}^0}_1^2$$
$${\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3-{\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1-{\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+{\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2-2\ {\mathbf{Y}^0}_1\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2+2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1)$$
$$\Omega^4+\big({\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_2^4+5\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2\ {\mathbf{Y}^0}_1^3-5\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1^4+{\mathbf{U}^0}_2^2\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2+{\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3-2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2$$
$${\mathbf{U}^0}_2+8\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_1^2+{\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2^2+2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3+2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3-2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1$$
$${\mathbf{U}^0}_2-2\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2-{\mathbf{U}^0}_1\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3-3\ {\mathbf{U}^0}_2\ {\mathbf{U}^0}_3\ {\mathbf{Y}^0}_2^4+4\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1+4\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_1^4\big)\ \Omega^2+{\mathbf{Y}^0}_1$$
$${\mathbf{U}^0}_3\ \big(-{\mathbf{Y}^0}_2^5\ {\mathbf{U}^0}_2-{\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+{\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^3+2\ {\mathbf{Y}^0}_1^5\ {\mathbf{U}^0}_2+2\ {\mathbf{Y}^0}_1^4\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2+{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_2-3\ {\mathbf{Y}^0}_1^2$$
$${\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_2-{\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^4\ {\mathbf{U}^0}_1-3\ {\mathbf{Y}^0}_1^4\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_1-5\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_1\big)\Big)\ {\mathbf{Y}^1}_1+\Big(-2\ {\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1^4+2\ {\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_2^4-2\ {\mathbf{Y}^0}_2$$
$${\mathbf{U}^0}_3^2\ {\mathbf{Y}^0}_1^3-2\ {\mathbf{U}^0}_1^2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3-2\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_2-2\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_1\ {\mathbf{U}^0}_2-2\ {\mathbf{U}^0}_2\ {\mathbf{Y}^0}_1^3\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_3-4\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1^2$$
$${\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2+2\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_3\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_1^2+2\ {\mathbf{Y}^0}_1^2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_2^2\ {\mathbf{U}^0}_2+2\ {\mathbf{U}^0}_1\ {\mathbf{Y}^0}_1\ {\mathbf{Y}^0}_2^3\ {\mathbf{U}^0}_3-{\mathbf{U}^0}_3\ {\mathbf{Y}^0}_1^2\ {\mathbf{Y}^0}_2\ {\mathbf{U}^0}_2^2+2\ {\mathbf{Y}^0}_1^3$$

$$
\mathbf{U}^0_1\,\mathbf{U}^0_2{}^2+\mathbf{U}^0_3\,\mathbf{Y}^0_1{}^3\,\mathbf{U}^0_2{}^2-\mathbf{Y}^0_1{}^3\,\mathbf{U}^0_1{}^2\,\mathbf{U}^0_3-4\,\mathbf{Y}^0_2\,\mathbf{U}^0_3\,\mathbf{U}^0_2\,\mathbf{Y}^0_1{}^2\,\mathbf{U}^0_1-2\,\mathbf{Y}^0_1\,\mathbf{Y}^0_2{}^2\,\mathbf{U}^0_3\,\mathbf{U}^0_2{}^2-\mathbf{Y}^0_1{}^2\,\mathbf{U}^0_1{}^2\,\mathbf{U}^0_3
$$

$$
\mathbf{Y}^0_2-2\,\mathbf{Y}^0_2{}^2\,\mathbf{U}^0_2{}^2\,\mathbf{U}^0_1\,\mathbf{Y}^0_1+\mathbf{Y}^0_2\,\mathbf{U}^0_2\,\mathbf{Y}^0_1{}^2\,\mathbf{U}^0_3{}^2+2\,\mathbf{Y}^0_2{}^2\,\mathbf{U}^0_1\,\mathbf{U}^0_3{}^2\,\mathbf{Y}^0_1+3\,\mathbf{Y}^0_2{}^2\,\mathbf{U}^0_2\,\mathbf{U}^0_3{}^2\,\mathbf{Y}^0_1-4\,\mathbf{Y}^0_2\,\mathbf{U}^0_1\,\mathbf{U}^0_3{}^2
$$

$$
\mathbf{Y}^0_1{}^2+2\,\mathbf{Y}^0_2{}^3\,\mathbf{U}^0_2\,\mathbf{U}^0_1{}^2-2\,\mathbf{Y}^0_1\,\mathbf{U}^0_1{}^3\,\mathbf{Y}^0_2{}^2+4\,\mathbf{U}^0_2\,\mathbf{U}^0_3{}^2\,\mathbf{Y}^0_1{}^3\Bigr)\,\Omega^3+\Bigl(-\mathbf{Y}^0_2\,(\mathbf{Y}^0_2+\mathbf{Y}^0_1)\,\mathbf{U}^0_3\,\Omega^5+\mathbf{Y}^0_2\,(-2\,\mathbf{Y}^0_1{}^2
$$

$$
\mathbf{Y}^0_2\,\mathbf{U}^0_2+\mathbf{Y}^0_1{}^3\,\mathbf{U}^0_3-2\,\mathbf{Y}^0_1{}^2\,\mathbf{Y}^0_2\,\mathbf{U}^0_1+2\,\mathbf{Y}^0_1\,\mathbf{Y}^0_2{}^2\,\mathbf{U}^0_1+3\,\mathbf{Y}^0_1{}^2\,\mathbf{Y}^0_2\,\mathbf{U}^0_3-\mathbf{Y}^0_1\,\mathbf{U}^0_3\,\mathbf{Y}^0_2{}^2+2\,\mathbf{U}^0_2\,\mathbf{Y}^0_1{}^3+\mathbf{Y}^0_2{}^3\,\mathbf{U}^0_3)
$$

$$
\Omega^3\Bigr)\,\mathbf{E}_3+\Bigl((\mathbf{Y}^0_1\,\mathbf{U}^0_1+\mathbf{Y}^0_2\,\mathbf{U}^0_2)\,\Omega^5-\mathbf{Y}^0_2{}^2\,(\mathbf{Y}^0_1\,\mathbf{U}^0_1+\mathbf{Y}^0_2\,\mathbf{U}^0_2)\,\Omega^3\Bigr)\,\mathbf{Y}^1_2{}^2+\Bigl(-\Omega^6\,\mathbf{Y}^0_1\,\mathbf{U}^0_3+\mathbf{Y}^0_1\,\mathbf{Y}^0_2\,(\mathbf{Y}^0_2+\mathbf{Y}^0_1)
$$

$$
\mathbf{U}^0_3\,\Omega^4\Bigr)\,\mathbf{U}^1_3+(-2\,\mathbf{Y}^0_1\,\mathbf{U}^0_1-2\,\mathbf{Y}^0_2\,\mathbf{U}^0_2)\,\Omega^7\,\frac{\partial\mathbf{E}_3}{\partial x_3}+\Bigl(\mathbf{Y}^0_1{}^2\,(-\mathbf{Y}^0_2+\mathbf{Y}^0_1)\,\mathbf{U}^0_3\,\Omega^4-\mathbf{Y}^0_1{}^2\,(\mathbf{Y}^0_2+\mathbf{Y}^0_1)\,(-\mathbf{Y}^0_2+\mathbf{Y}^0_1)^2
$$

$$
\mathbf{U}^0_3\,\Omega^2\Bigr)\,\mathbf{U}^1_1+\Bigl(-4\,\Omega^7\,\mathbf{Y}^0_2+(2\,\mathbf{Y}^0_1{}^2\,\mathbf{U}^0_3+12\,\mathbf{U}^0_1\,\mathbf{Y}^0_2{}^2-12\,\mathbf{Y}^0_2\,\mathbf{U}^0_2\,\mathbf{Y}^0_1)\,\Omega^5-2\,\mathbf{Y}^0_1\,\mathbf{Y}^0_2\,(\mathbf{Y}^0_2+\mathbf{Y}^0_1)\,(\mathbf{Y}^0_1
$$

$$
\mathbf{U}^0_3+\mathbf{U}^0_2\,\mathbf{Y}^0_1-\mathbf{U}^0_1\,\mathbf{Y}^0_2)\,\Omega^3+2\,\mathbf{Y}^0_1\,\mathbf{Y}^0_2\,(\mathbf{Y}^0_1{}^3+2\,\mathbf{Y}^0_2{}^3+\mathbf{Y}^0_2\,\mathbf{Y}^0_1{}^2)\,(-\mathbf{U}^0_1\,\mathbf{Y}^0_2+\mathbf{U}^0_2\,\mathbf{Y}^0_1)\,\Omega\Bigr)\,\mathbf{E}_1+(2\,\mathbf{Y}^0_2
$$

$$
\mathbf{U}^0_2+2\,\mathbf{Y}^0_1\,\mathbf{U}^0_1)\,\Omega^7+\Bigl(\mathbf{Y}^0_1{}^2\,(\mathbf{Y}^0_2+\mathbf{Y}^0_1)\,\mathbf{U}^0_3\,\Omega^4-\mathbf{Y}^0_1\,(\mathbf{Y}^0_2+\mathbf{Y}^0_1)\,(\mathbf{Y}^0_2{}^3-\mathbf{Y}^0_1\,\mathbf{Y}^0_2{}^2+\mathbf{Y}^0_2\,\mathbf{Y}^0_1{}^2+\mathbf{Y}^0_1{}^3)\,\mathbf{U}^0_3\,\Omega^2\Bigr)
$$

$$
\mathbf{U}^1_2+\Bigl((\mathbf{Y}^0_2\,\mathbf{U}^0_2+4\,\mathbf{Y}^0_1\,\mathbf{U}^0_1+2\,\mathbf{U}^0_1\,\mathbf{Y}^0_2)\,\Omega^5+(-4\,\mathbf{Y}^0_1{}^3\,\mathbf{U}^0_1-\mathbf{Y}^0_2{}^3\,\mathbf{U}^0_1-2\,\mathbf{Y}^0_2{}^3\,\mathbf{U}^0_2-6\,\mathbf{Y}^0_1\,\mathbf{Y}^0_2{}^2\,\mathbf{U}^0_1+\mathbf{Y}^0_1
$$

$$
\mathbf{Y}^0_2{}^2\,\mathbf{U}^0_2-\mathbf{Y}^0_1{}^2\,\mathbf{Y}^0_2\,\mathbf{U}^0_1-\mathbf{Y}^0_1{}^2\,\mathbf{Y}^0_2\,\mathbf{U}^0_2)\,\Omega^3+\mathbf{Y}^0_1\,\mathbf{Y}^0_2\,(\mathbf{Y}^0_1{}^2+2\,\mathbf{Y}^0_1\,\mathbf{Y}^0_2-\mathbf{Y}^0_2{}^2)\,(\mathbf{Y}^0_1\,\mathbf{U}^0_1+\mathbf{Y}^0_2\,\mathbf{U}^0_2)\,\Omega\Bigr)
$$

$$
\mathbf{Y}^1_1{}^2\Bigr)\Big/\,\Omega^8,
$$

$$
\mathbf{U}^1(s;\mathbf{x},\mathbf{v},s)=0.
$$

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